Weighted projective lines and invariant flags of nilpotent operators

Dirk Kussin

Tokyo, 15 August, 2010

Birkhoff type problems

Definition (Homomorphism categories)

Let Λ be an Artin algebra. Denote by $\mathcal{H}(\Lambda)$ the category of all morphisms

$$A \xrightarrow{f} B$$

in $\operatorname{mod}\Lambda$. Morphisms in $\mathcal{H}(\Lambda)$ are given by commutative diagrams.

Remark

$\mathcal{H}(\Lambda) \simeq \operatorname{mod}\begin{pmatrix}\Lambda & \Lambda \\ 0 & \Lambda\end{pmatrix}$ is abelian.

Definition (Submodule categories)

Let $\mathfrak{S}(\Lambda) \subseteq \mathcal{H}(\Lambda)$ be the full subcategory of monomorphisms (embeddings).

Remark

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Classify the objects in $\mathfrak{S}(\Lambda)$.

Examples

1. G. Birkhoff (1934): Subgroups of finite abelian p-groups (p prime). $\Lambda = \mathbb{Z}/p^b$. In 1999 classificication was done by Richman-Walker for $b \leq 5$.
2. C. M. Ringel and M. Schmidmeier (2006, 2008): Invariant subspaces of nilpotent operators. $\Lambda = k[T]/T^b$ (b fixed nilpotency degree).
3. D. Simson (2002, 2007): More complex subconfigurations than in 2. Birkhoff type problems.

Several other authors working on this subject: Audrey Moore, Pu Zhang, Xiao-Wu Chen, ...

Alternative approach: K-Lenzing-Meltzer

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$\mathfrak{S}(b)$: Category of triples (V, f, U), with V fin. dim. k-vector space, f an endomorphism of V such that $f^b = 0$ and $U \subseteq V$ is an f-invariant subspace. Then $\mathfrak{S}(b) \simeq \mathfrak{S}(k[T]/T^b)$.

Ringel-Schmidmeier (2008)

Construction of AR-quivers of $\mathfrak{S}(b)$.

1. *For $b \leq 5$ is $\mathfrak{S}(b)$ of finite representation type.*
2. *$\mathfrak{S}(6)$ is tame of tubular type.*
3. *For $b \geq 7$ is $\mathfrak{S}(b)$ of wild type.*

Tool: Covering situation. $k[T]/T^b$ is $\mathbb{Z}$-graded by $\deg T = 1$. Considering graded submodules, morphisms of degree zero, yields $\mathfrak{S}^{\mathbb{Z}}(b)$.

Proposition (Ringel-Schmidmeier (2008))

The natural covering functor $\mathfrak{S}^{\mathbb{Z}}(b) \to \mathfrak{S}(b)$ is dense for $b \leq 6$.

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Ringel-Schmidmeier observed that the stable category

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is (triangulated) Calabi-Yau and computed the (fractional) Calabi-Yau dimension.

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More general setting considered by D. Simson (2002, 2007):

Let $(I, \leq)$ be a finite poset with unique maximal element $*$.

Denote by $\mathfrak{S}_I(b)$ be the category of tuples $(U_i)_{i\in I}$ such that

1. U_i is a fin. dim. $k[T]/T^b$-module for each $i \in I$;
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Similarly $\mathfrak{S}_I^{\mathbb{Z}}(b)$ is defined in the graded sense.

Consider I as quiver by $i \to j$ for $j \leq i$.

Let $\mathcal{A}_I^{(\mathbb{Z})}(b)$ be the category of (contravariant) representations of I in $\mathrm{mod}^{(\mathbb{Z})}(k[T]/T^b)$.

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$\mathfrak{S}_I^{(\mathbb{Z})}(b)$ is equivalent to the full subcategory of $\mathcal{A}_I^{(\mathbb{Z})}(b)$ consisting of the mono representations, that is, where all arrows of I are realized by mono's.

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Note

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One flag of length a. $I : * \to 1 \to 2 \to \cdots \to a-2 \to a-1$.

$\mathfrak{S}_a(b) := \mathfrak{S}_I(b)$, $\mathfrak{S}_a^{\mathbb{Z}}(b) := \mathfrak{S}_I^{\mathbb{Z}}(b)$.

Two flags of lengths a_1, a_2, resp. I:

$$\begin{array}{l} a_2 - 1 \\ \uparrow \\ a_2 - 2 \\ \uparrow \\ \vdots \\ \uparrow \\ 1 \\ \uparrow \\ * \longrightarrow 1 \to \cdots \to a_1 - 2 \to a_1 - 1 \end{array}$$

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Weighted projective lines with triple weights

Integers $a_1,\ a_2,\ b \geq 2$. Wlog $a_1 \leq a_2$.

$$S = \frac{k[X_1, X_2, Y]}{X_1^{a_1} + X_2^{a_2} + Y^b}$$ commutative ring

$\mathbb{L} = \mathbb{L}(a_1, a_2, b) = \langle \vec{x}_1, \vec{x}_2, \vec{y} \rangle$ abelian group of rank 1 with relations

$$a_1\vec{x}_1 = a_2\vec{x}_2 = b\vec{y}.$$

S is $\mathbb{L}$-graded by $\deg X_i = \vec{x}_i$, $\deg Y = \vec{y}$.

$\mathbb{X} = \mathbb{X}(a_1, a_2, b)$ weighted projective line (Geigle-Lenzing)

category of (graded) coherent sheaves $\operatorname{coh}(\mathbb{X}) = \operatorname{mod}^{\mathbb{L}}(S)/\operatorname{mod}_0^{\mathbb{L}}(S)$

induced quotient functor $q\colon \operatorname{mod}^{\mathbb{L}}(S) \to \operatorname{coh}(\mathbb{X})$ sends $S(\vec{x})$ to $\mathcal{O}(\vec{x})$.

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- *is a Hom-finite hereditary category*
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- *has Serre duality* $\operatorname{D}\operatorname{Ext}^1(X, Y) = \operatorname{Hom}(Y, X(\vec{\omega}))$ *with* $\vec{\omega} = -\vec{x}_1 - \vec{x}_2 + (b-1)\vec{y}$ *the dualizing element*
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Vector bundles

$\mathrm{vect}(\mathbb{X}) \subseteq \mathrm{coh}(\mathbb{X})$ full subcategory of vector bundles ($=$ torsion-free sheaves).

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Lemma ($\vec{y}$-normalform)

Each $\vec{x} \in \mathbb{L}$ has a unique expression

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with $0 \leq m_i \leq a_i - 1$ and $n \in \mathbb{Z}$.

$\Rightarrow$ Fundamentaldomain of $\mathbb{L}$ mod $\mathbb{Z}\vec{y}$, e.g. in case $(5, 4, b)$:

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   $\mathcal{F} = \mathcal{L} \setminus \mathcal{P}$. Call a line bundle $L \in \mathcal{P}$ *persistent*, $L \in \mathcal{F}$ *fading*.

2. $\mathcal{P}^{proj} := \{\mathcal{O}(\vec{x}) \in \mathcal{P} \mid m_1 = 0 \text{ or } m_2 = 0\}$. Call $L \in \mathcal{P}^{proj}$ a *projective* line bundle.
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Example: Case $(5, 4, b)$.

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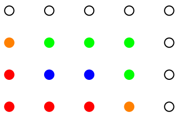

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AR translation τ is given by shift by

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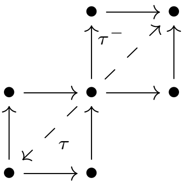

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Main results

Joint work with Helmut Lenzing and Hagen Meltzer.

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1. *The equivalence*

$$\mathrm{vect}(\mathbb{X}) \simeq \mathrm{CM}^{\mathbb{L}}(S) \subseteq \mathrm{mod}^{\mathbb{L}}(S)$$

   induces an exact structure on $\mathrm{vect}(\mathbb{X})$*, so that* $\mathrm{vect}(\mathbb{X})$ *is a Frobenius category with* $\mathcal{L}$ *the system of indecomposable projectives/injectives.*

2. *A sequence* $\eta\colon 0 \to A \to B \to C \to 0$ *in* $\mathrm{vect}(\mathbb{X})$ *is exact if and only if* $\mathrm{Hom}(L, \eta)$ *is exact for all* $L \in \mathcal{L}$*. "*$\mathcal{L}$*-exactness"*

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Theorem A (KLM 2010)

On the factor category

$$\mathrm{vect}(\mathbb{X})/[\mathcal{F}]$$

there are two natural exact structures:

1. *one induced by $\mathcal{L}$-exactness such that it becomes a Frobenius category with indec. projectives/injectives given by $\mathcal{P}$.*
2. *the other induced by $\partial\mathcal{L}$-exactness, such that the indec. projectives given by $\mathcal{P}^{proj}$ and indec. injectives given by $\mathcal{P}^{inj}$.*

Moreover, these two structures coincide if and only if $a_1 = 2$, i.e. in the one-flag case.

We call $\mathrm{vect}(\mathbb{X})/[\mathcal{F}]$ with the exact structure induced by $\partial\mathcal{L}$-exactness *almost Frobenius*. One has to enlarge the system of projectives by forming finite segments in order to get a Frobenius category:

$$\mathcal{P} = \sqcup_{P \in \mathcal{P}^{proj}} \{P, \tau^{-}P, \ldots, \tau^{-n_P}P\}$$

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Theorem A (KLM 2010)

On the factor category

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there are two natural exact structures:

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$\mathcal{P} = \mathcal{P}^{proj} \quad \Leftrightarrow \quad \mathcal{P}^{proj} = \mathcal{P}^{inj} \quad \Leftrightarrow \quad a_1 = 2 \quad \Leftrightarrow \quad$ one-flag case.

Proof immediately clear from the picture

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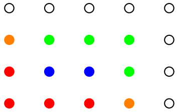

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$\mathbb{X} = \mathbb{X}(a_1, a_2, b)$.

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$$\text{vect}(\mathbb{X})/[\mathcal{F}] \simeq \mathfrak{S}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$$

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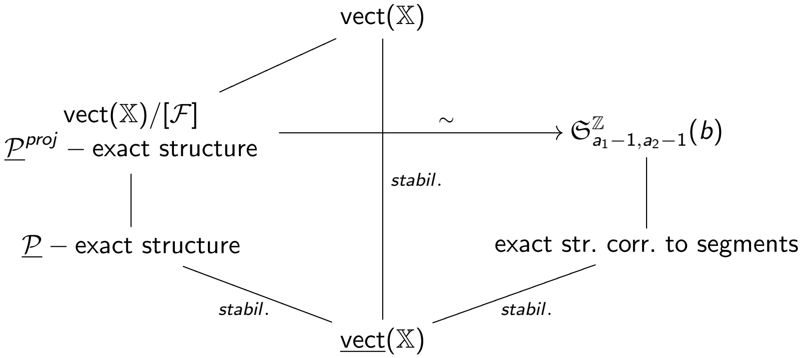
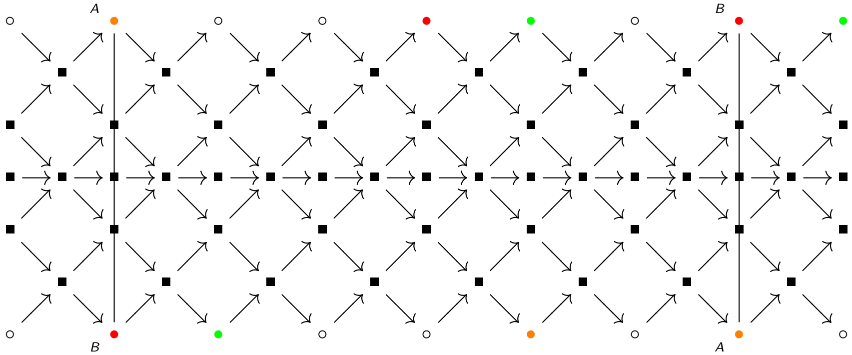
Schematic overview

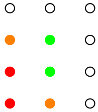


Examples

$(3, 4, 2)$: a proper 2-flag case



Fundamental domain mod $\mathbb{Z}\vec{y}$



One weight type / three different flag problems

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This also explains a duality effect observed by Happel-Seidel.

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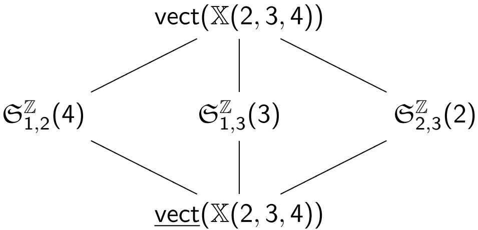

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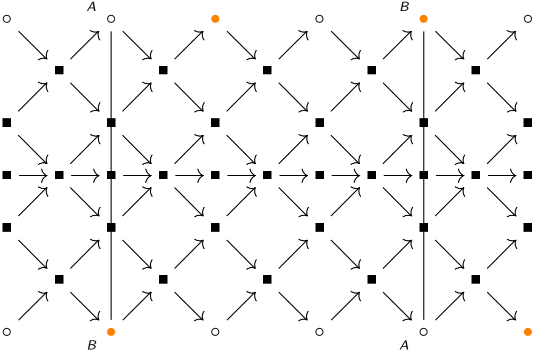
Case $(2, 3, 4)$



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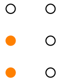
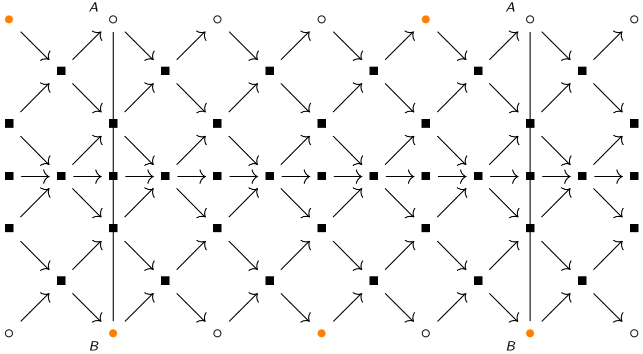


Case $(2, 4, 3)$



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