# Weighted projective lines and invariant flags of nilpotent operators

Dirk Kussin

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### Definition (Homomorphism categories)

Let $\Lambda$ be an Artin algebra. Denote by $\mathcal{H}(\Lambda)$ the category of all morphisms

$$A \xrightarrow{f} B$$

in $\operatorname{mod}\Lambda$. Morphisms in $\mathcal{H}(\Lambda)$ are given by commutative diagrams.

### Remark

$\mathcal{H}(\Lambda) \simeq \operatorname{mod}\begin{pmatrix}\Lambda & \Lambda \\ 0 & \Lambda\end{pmatrix}$ is abelian.

### Definition (Submodule categories)

Let $\mathfrak{S}(\Lambda) \subseteq \mathcal{H}(\Lambda)$ be the full subcategory of monomorphisms (embeddings).

### Remark

$\mathfrak{S}(\Lambda)$ is an exact Krull-Remak-Schmidt category (induced by structure on $\mathcal{H}(\Lambda)$).

## Birkhoff Problem

Classify the objects in $\mathfrak{S}(\Lambda)$.

## Examples

1. G. Birkhoff (1934): Subgroups of finite abelian $p$-groups ($p$ prime). $\Lambda = \mathbb{Z}/p^b$. In 1999 classificication was done by Richman-Walker for $b \leq 5$.
2. C. M. Ringel and M. Schmidmeier (2006, 2008): Invariant subspaces of nilpotent operators. $\Lambda = k[T]/T^b$ ($b$ fixed nilpotency degree).
3. D. Simson (2002, 2007): More complex subconfigurations than in 2. Birkhoff type problems.

Several other authors working on this subject: Audrey Moore, Pu Zhang, Xiao-Wu Chen, ...

Alternative approach: K-Lenzing-Meltzer

Fix a field $k$ and an integer $b \geq 2$.

$\mathfrak{S}(b)$: Category of triples $(V, f, U)$, with $V$ fin. dim. $k$-vector space, $f$ an endomorphism of $V$ such that $f^b = 0$ and $U \subseteq V$ is an $f$-invariant subspace. Then $\mathfrak{S}(b) \simeq \mathfrak{S}\big(k[T]/T^b\big)$.

### Ringel-Schmidmeier (2008)

*Construction of AR-quivers of $\mathfrak{S}(b)$.*

1. *For $b \leq 5$ is $\mathfrak{S}(b)$ of finite representation type.*
2. *$\mathfrak{S}(6)$ is tame of tubular type.*
3. *For $b \geq 7$ is $\mathfrak{S}(b)$ of wild type.*

**Tool:** Covering situation. $k[T]/T^b$ is $\mathbb{Z}$-graded by $\deg T = 1$. Considering graded submodules, morphisms of degree zero, yields $\mathfrak{S}^{\mathbb{Z}}(b)$.

### Proposition (Ringel-Schmidmeier (2008))

*The natural covering functor $\mathfrak{S}^{\mathbb{Z}}(b) \to \mathfrak{S}(b)$ is dense for $b \leq 6$.*

# Frobenius category and stable category

**Lemma**

$\mathfrak{S}(b)$ *is a Frobenius category.*

Ringel-Schmidmeier observed that the stable category

$$\underline{\mathfrak{S}}(b)$$

is (triangulated) Calabi-Yau and computed the (fractional) Calabi-Yau dimension.

Moreover: these categories form a so-called ADE-chain (compare Lenzing's workshop talk).

More general setting considered by D. Simson (2002, 2007):

Let $(I, \leq)$ be a finite poset with unique maximal element $*$.

Denote by $\mathfrak{S}_I(b)$ be the category of tuples $(U_i)_{i \in I}$ such that

1. $U_i$ is a fin. dim. $k[T]/T^b$-module for each $i \in I$;
2. $U_i \subseteq U_j$ is a submodule for all $i \leq j$.

Similarly $\mathfrak{S}_I^{\mathbb{Z}}(b)$ is defined in the graded sense.

Consider $I$ as quiver by $i \to j$ for $j \leq i$.

Let $\mathcal{A}_I^{(\mathbb{Z})}(b)$ be the category of (contravariant) representations of $I$ in $\text{mod}^{(\mathbb{Z})}(k[T]/T^b)$.

**Lemma**

*$\mathfrak{S}_I^{(\mathbb{Z})}(b)$ is equivalent to the full subcategory of $\mathcal{A}_I^{(\mathbb{Z})}(b)$ consisting of the mono representations, that is, where all arrows of $I$ are realized by mono's.*

### Corollary

*$\mathfrak{S}_I(b)$ and $\mathfrak{S}_I^{\mathbb{Z}}(b)$ are exact categories (Quillen sense) with enough projectives and enough injectives.*

### Note

In general $\mathfrak{S}_I(b)$ and $\mathfrak{S}_I^{\mathbb{Z}}(b)$ are *not* Frobenius!

### Simson (2002, 2007)

*Determination of the representation type of $\mathfrak{S}_I(b)$.*

# Flags

One flag of length $a$. $I : * \to 1 \to 2 \to \cdots \to a-2 \to a-1$.

$\mathfrak{S}_a(b) := \mathfrak{S}_I(b)$, $\mathfrak{S}_a^{\mathbb{Z}}(b) := \mathfrak{S}_I^{\mathbb{Z}}(b)$.

Two flags of lengths $a_1$, $a_2$, resp. $I$:

$$\begin{array}{l}
a_2 - 1 \\
\uparrow \\
a_2 - 2 \\
\uparrow \\
\vdots \\
\uparrow \\
1 \\
\uparrow \\
* \longrightarrow 1 \to \ldots \to a_1 - 2 \to a_1 - 1
\end{array}$$

$\mathfrak{S}_{a_1,a_2}(b) := \mathfrak{S}_I(b)$, $\mathfrak{S}_{a_1,a_2}^{\mathbb{Z}}(b) := \mathfrak{S}_I^{\mathbb{Z}}(b)$.

Ringel-Schmidmeier: $\mathfrak{S}_{1,2}(b)$. Moore-Schmidmeier: e.g. tubular case $\mathfrak{S}_{1,3}(4)$, ...

Integers $a_1, a_2, b \geq 2$. Wlog $a_1 \leq a_2$.

$$S = \frac{k[X_1, X_2, Y]}{X_1^{a_1} + X_2^{a_2} + Y^b} \text{ commutative ring}$$

$\mathbb{L} = \mathbb{L}(a_1, a_2, b) = \langle \vec{x}_1, \vec{x}_2, \vec{y} \rangle$ abelian group of rank 1 with relations

$$a_1\vec{x}_1 = a_2\vec{x}_2 = b\vec{y}.$$

$S$ is $\mathbb{L}$-graded by $\deg X_i = \vec{x}_i$, $\deg Y = \vec{y}$.

$\mathbb{X} = \mathbb{X}(a_1, a_2, b)$ weighted projective line (Geigle-Lenzing)

category of (graded) coherent sheaves $\mathrm{coh}(\mathbb{X}) = \mathrm{mod}^{\mathbb{L}}(S)/\mathrm{mod}_0^{\mathbb{L}}(S)$

induced quotient functor $q\colon \mathrm{mod}^{\mathbb{L}}(S) \to \mathrm{coh}(\mathbb{X})$ sends $S(\vec{x})$ to $\mathcal{O}(\vec{x})$.

### Proposition

$\mathrm{coh}(\mathbb{X})$

- *is a Hom-finite hereditary category*
- *is a Krull-Remak-Schmidt category*
- *has Serre duality* $\mathrm{D}\,\mathrm{Ext}^1(X, Y) = \mathrm{Hom}(Y, X(\vec{\omega}))$ *with* $\vec{\omega} = -\vec{x}_1 - \vec{x}_2 + (b-1)\vec{y}$ *the dualizing element*
- *has a tilting object.*

# Vector bundles

$\mathrm{vect}(\mathbb{X}) \subseteq \mathrm{coh}(\mathbb{X})$ full subcategory of vector bundles
(= torsion-free sheaves).

## Proposition

*The quotient functor $q$ induces an equivalence*

$$\mathrm{CM}^{\mathbb{L}}(S) \simeq \mathrm{vect}(\mathbb{X}).$$

## Proposition

1. *Each line bundle is isomorphic to $\mathcal{O}(\vec{x})$ for some unique $\vec{x} \in \mathbb{L}$. Thus*

$$\mathcal{L} := \mathrm{Pic}\,\mathbb{X} \simeq \mathbb{L}.$$

2. $\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \simeq S_{\vec{y}-\vec{x}}$.

## Lemma ($\vec{y}$-normalform)

*Each $\vec{x} \in \mathbb{L}$ has a unique expression*

$$\vec{x} = m_1\vec{x}_1 + m_2\vec{x}_2 + n\vec{y}$$

*with $0 \leq m_i \leq a_i - 1$ and $n \in \mathbb{Z}$.*

$\Rightarrow$ Fundamentaldomain of $\mathbb{L}$ mod $\mathbb{Z}\vec{y}$, e.g. in case $(5, 4, b)$:

$$\begin{array}{ccccccccc}
3\vec{x}_2 & \longrightarrow & \vec{x}_1 + 3\vec{x}_2 & \longrightarrow & 2\vec{x}_1 + 3\vec{x}_2 & \longrightarrow & 3\vec{x}_1 + 3\vec{x}_2 & \longrightarrow & 4\vec{x}_1 + 3\vec{x}_2 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
2\vec{x}_2 & \longrightarrow & \vec{x}_1 + 2\vec{x}_2 & \longrightarrow & 2\vec{x}_1 + 2\vec{x}_2 & \longrightarrow & 3\vec{x}_1 + 2\vec{x}_2 & \longrightarrow & 4\vec{x}_1 + 2\vec{x}_2 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vec{x}_2 & \longrightarrow & \vec{x}_1 + \vec{x}_2 & \longrightarrow & 2\vec{x}_1 + \vec{x}_2 & \longrightarrow & 3\vec{x}_1 + \vec{x}_2 & \longrightarrow & 4\vec{x}_1 + \vec{x}_2 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vec{0} & \longrightarrow & \vec{x}_1 & \longrightarrow & 2\vec{x}_1 & \longrightarrow & 3\vec{x}_1 & \longrightarrow & 4\vec{x}_1
\end{array}$$

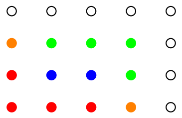
## Definition

1. $\mathcal{L} = \mathcal{P} \sqcup \mathcal{F}$ with

$$\mathcal{P} = \{\mathcal{O}(\vec{x}) \mid \vec{x} = m_1\vec{x}_1 + m_2\vec{x}_2 + n\vec{y},\ 0 \leq m_i \leq a_i - 2,\ n \in \mathbb{Z}\},$$

   $\mathcal{F} = \mathcal{L} \setminus \mathcal{P}$. Call a line bundle $L \in \mathcal{P}$ *persistent*, $L \in \mathcal{F}$ *fading*.
2. $\mathcal{P}^{proj} := \{\mathcal{O}(\vec{x}) \in \mathcal{P} \mid m_1 = 0 \text{ or } m_2 = 0\}$. Call $L \in \mathcal{P}^{proj}$ a *projective* line bundle.
3. $\mathcal{P}^{inj} := \{\mathcal{O}(\vec{x}) \in \mathcal{P} \mid m_1 = a_1 - 2 \text{ or } m_2 = a_2 - 2\}$. Call $L \in \mathcal{P}^{inj}$ an *injective* line bundle.
4. $\partial\mathcal{L} := \mathcal{P}^{proj} \sqcup \mathcal{F}$.

Example: Case $(5, 4, b)$.

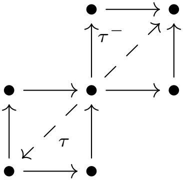


$\circ = \mathcal{F}$, $\bullet$ (red) $= \mathcal{P}^{proj} \setminus \mathcal{P}^{inj}$, $\bullet$ (green) $= \mathcal{P}^{inj} \setminus \mathcal{P}^{proj}$, $\bullet$ (orange) $= \mathcal{P}^{proj} \cap \mathcal{P}^{inj}$

# Auslander-Reiten translation

AR translation $\tau$ is given by shift by

$$\vec{\omega} = -\vec{x}_1 - \vec{x}_2 + (b-1)\vec{y} \equiv -\vec{x}_1 - \vec{x}_2 \bmod \mathbb{Z}\vec{y}.$$




If getting out of the fundamental region compute modulo $a_1\vec{x}_1$ or modulo $a_2\vec{x}_2$, resp.

Joint work with Helmut Lenzing and Hagen Meltzer.

## Proposition

1. *The equivalence*

$$\mathrm{vect}(\mathbb{X}) \simeq \mathrm{CM}^{\mathbb{L}}(S) \subseteq \mathrm{mod}^{\mathbb{L}}(S)$$

   *induces an exact structure on* $\mathrm{vect}(\mathbb{X})$*, so that* $\mathrm{vect}(\mathbb{X})$ *is a Frobenius category with* $\mathcal{L}$ *the system of indecomposable projectives/injectives.*
2. *A sequence* $\eta\colon 0 \to A \to B \to C \to 0$ *in* $\mathrm{vect}(\mathbb{X})$ *is exact if and only if* $\mathrm{Hom}(L, \eta)$ *is exact for all* $L \in \mathcal{L}$*. "*$\mathcal{L}$*-exactness"*

Similarly one defines $\partial\mathcal{L}$-exactness, which provides an additional exact structure on $\mathrm{vect}(\mathbb{X})$.

## Theorem A (KLM 2010)

*On the factor category*

$$\mathrm{vect}(\mathbb{X})/[\mathcal{F}]$$

*there are two natural exact structures:*

1. *one induced by $\mathcal{L}$-exactness such that it becomes a Frobenius category with indec. projectives/injectives given by $\mathcal{P}$.*
2. *the other induced by $\partial\mathcal{L}$-exactness, such that the indec. projectives given by $\mathcal{P}^{proj}$ and indec. injectives given by $\mathcal{P}^{inj}$.*

*Moreover, these two structures coincide if and only if $a_1 = 2$, i.e. in the one-flag case.*

We call $\mathrm{vect}(\mathbb{X})/[\mathcal{F}]$ with the exact structure induced by $\partial\mathcal{L}$-exactness *almost Frobenius*. One has to enlarge the system of projectives by forming finite segments in order to get a Frobenius category:

$$\mathcal{P} = \sqcup_{P \in \mathcal{P}^{proj}} \{P, \tau^{-}P, \ldots, \tau^{-n_P}P\}$$

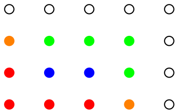
with $\tau^{-n_P}P$ injective. $P \mapsto \tau^{-n_P}P$ induces a bijective correspondence between projectives and injectives.

# Coincidence of exact structures

## Lemma

$\mathcal{P} = \mathcal{P}^{proj} \quad \Leftrightarrow \quad \mathcal{P}^{proj} = \mathcal{P}^{inj} \quad \Leftrightarrow \quad a_1 = 2 \quad \Leftrightarrow$ *one-flag case.*

**Proof** immediately clear from the picture



## Theorem B (KLM 2010)

$\mathbb{X} = \mathbb{X}(a_1, a_2, b)$.

1. *The functor* $E \mapsto \underline{\mathcal{P}}^{proj}(-, E)$ *induces an equivalence*

$$\mathrm{vect}(\mathbb{X})/[\mathcal{F}] \simeq \mathfrak{S}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$$

   *of almost Frobenius categories. Shift by* $\vec{y}$ *corresponds to degree shift by* 1.

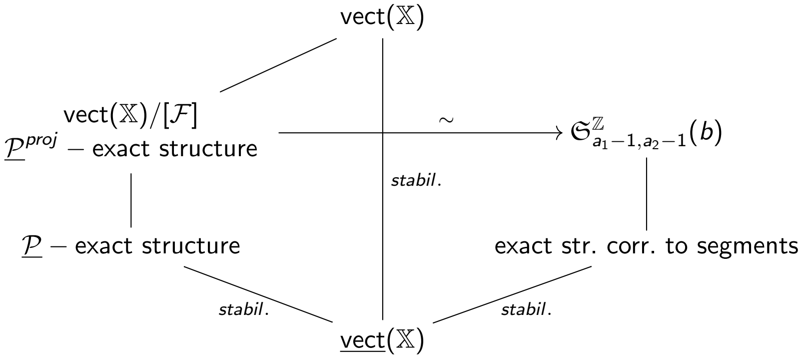
2. *The stable categories* $\big(\mathrm{vect}(\mathbb{X})/[\mathcal{F}]\big)/\big(\mathcal{P}/[\mathcal{F}]\big)$ *and* $\mathrm{vect}(\mathbb{X})/[\mathcal{L}]$ *are equivalent as triangulated categories; notation:* $\underline{\mathrm{vect}}(\mathbb{X})$.

## Applications

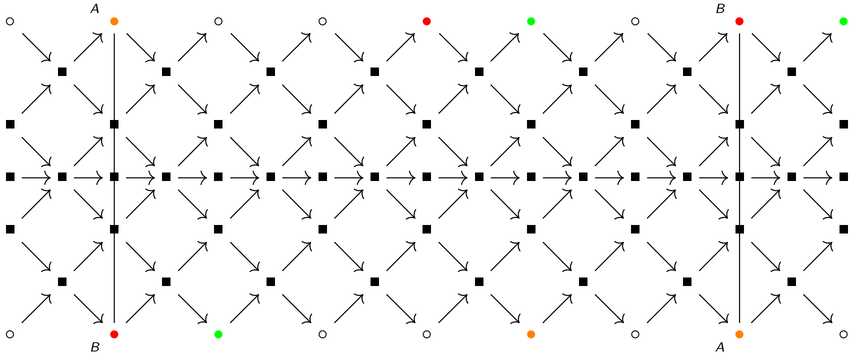
1. AR-quiver of $\mathfrak{S}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$ and $\underline{\mathfrak{S}}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$ by deleting (fading, all, resp.) line bundles from AR-quiver of $\mathrm{vect}(\mathbb{X})$.
2. Tilting objects in $\underline{\mathfrak{S}}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$.
3. Fractional CY-dimension of $\underline{\mathfrak{S}}^{\mathbb{Z}}_{a_1-1,a_2-1}(b)$ from Euler characteristic.

For more details compare Meltzer's talk this afternoon.

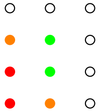
# Schematic overview



# $(3,4,2)$: a proper 2-flag case

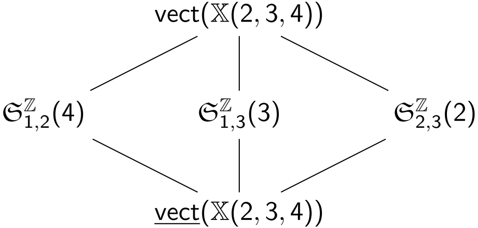



Fundamental domain mod $\mathbb{Z}\vec{y}$



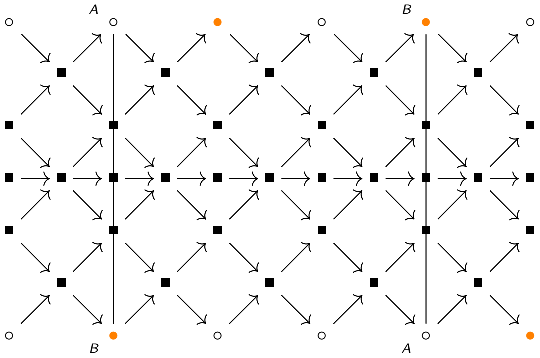
# One weight type / three different flag problems

If the three weights are pairwise different, e.g. $(2,3,4)$, we get three different systems of fading line bundles and hence three different factor categories:

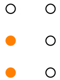



This also explains a duality effect observed by Happel-Seidel.

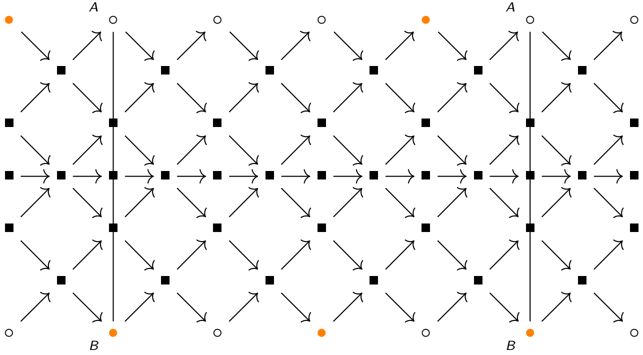
# Case $(2, 3, 4)$




Fundamental domain mod $\mathbb{Z}\vec{y}$



# Case $(2, 4, 3)$




Fundamental domain mod $\mathbb{Z}\vec{y}$