Transfer operators for hard rod type models

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Abstract

We describe a new approach to the hard rod model and construct a Ruelle-Mayer type transfer operator which satisfies a dynamical trace formula. This yields a meromorphic continuation of the dynamical zeta function in two variables among them the inverse temperature.

Keywords: Hard rod model, spin chain, transfer operator, trace formulae, dynamical zeta function.

1 Introduction

The hard rod models (HRM) describe one-dimensional particles moving along the real line under the influence of a two-body interaction. The particles have a positive finite diameter and are assumed to be solid (hard), thus cannot overlap each other. Usually ([3], [7], [8]), the HRM is modelled as particle system with a continuous configuration space. We introduce a new approach to the HRM, namely we give a modelling as a one-dimensional one-sided subshift, i.e., as a spin chain, where the spin variable is understood as the location of a rod relative to a lattice point. Our modelling enables us to use the machinery developed in [5] for two special classes of long-range interactions. These contain exponentially decaying interactions. We construct a transfer operator and prove a dynamical trace formula for the sequence \((Z_n^p(\beta))_{n \in \mathbb{N}}\) of partition functions depending on a parameter \(\beta \in \mathbb{C}\), the (complexified) inverse temperature. Using [5] we show that Ruelle’s dynamical zeta function ([11], [12])

\[
\zeta_R(z; \beta) := \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta) \right)
\]

can be extended meromorphically to \((z, \beta) \in \mathbb{C}^2\). In order to apply the transfer operator method for long-range interactions, one has to work with periodic boundary conditions, [7] or [10, 1.3]. The partition function \(Z_n^p\) for our hard rod type model is defined by the interaction energy of \(n\) rods of diameter \(0 < a \leq b\) inside the box \([0, nb]\) and their interactions with the periodically extended spin configuration outside the box. We mention the first transfer operator approach for the HRM by D. Mayer and K. Viswanathan in [8] who studied purely exponentially decaying interactions. They are using the grand canonical partition function \(Z_n\) for the system with periodic boundary conditions. By long and explicit calculations they derived a different transfer operator and a different trace formula which did not allow them to show that the associated dynamical zeta function has a meromorphic extension except for vanishing interaction.

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2 Hard Rod Models

We define a hard rod model or hard rod subshift \((F, \nu, A_{a,b}, A_{(\Delta)})\) to be a spin chain, cf. [13] or [10], described by the following data: Let \(F \subset \mathbb{R}\) be a (not necessarily bounded) interval containing zero equipped with a finite measure \(\nu\). For \(a, b > 0\) we define the transition function

\[
A_{a,b} : F \times F \to \{0, 1\}, \ (x, y) \mapsto A_{a,b}(x, y) = \begin{cases} 1, & \text{if } y + b - x \geq a, \\ 0, & \text{otherwise}. \end{cases}
\]

The left shift \(\tau : F^N \to F^N, \ (\tau \xi)_k := \xi_{k+1}\) leaves the configuration space

\[
\Omega := \{\xi \in F^N | A_{a,b}(\xi_i, \xi_{i+1}) = 1 \ \forall i\}
\]

invariant. With an observable, i.e., a continuous bounded function \(A : \Omega \to \mathbb{C}\), we associate a partition function

\[
Z_n(A) := \int_{F^n} \prod_{i=1}^n A_{a,b}(\xi_i, \xi_{i+1}) \exp \left( \sum_{k=0}^{n-1} A(\tau^k(\xi_1, \ldots, \xi_n)) \right) d\nu^N(\xi_1, \ldots, \xi_n),
\]

where \(\xi_1 \ldots \xi_n := (\xi_1, \ldots, \xi_n, \xi_1, \ldots, \xi_n, \ldots)\) and \(\xi_{n+1} := \xi_1\). We will consider observables of the form

\[
A_{(\Delta)} : \Omega \to \mathbb{C}, \ \xi \mapsto \sum_{i=2}^{\infty} \Delta(\xi_i - \xi_1 + (i - 1)b)
\]

for a given function \(\Delta : \mathbb{R} \to \mathbb{C}\) with a suitable decay ensuring that the above series converges. We interpret \(a\) as the length of the hard rods, \(A_{(\Delta)}(\xi)\) as the sum of two-body interactions between the first particle and the rest of the half line, and \(F\) as the set of possible locations of a rod relative to the lattice points as we will explain next.

**Proposition 2.1.** Let \((F, \nu, A_{a,b}, A_{(\Delta)})\) be a hard rod subshift and \(p : F^N \to \mathbb{R}^N, \ (\xi_i)_{i \in \mathbb{N}} \mapsto (ib + \xi_i)_{i \in \mathbb{N}}\) the so called position map.

(i) Let \(\xi \in F^N\), then \(\xi\) belongs to \(\Omega\) iff \(p(\xi_{i+1}) - p(\xi_i) \geq a\) for all \(i \in \mathbb{N}\).

(ii) Let \(\xi \in \Omega\) and \(i \in \mathbb{N}\), then \(\xi_i \geq \xi_1 + (i - 1)(a - b)\).

(iii) If \(a > b\), then \(\Omega\) does not contain any \(\tau^n\)-periodic sequences.

![Figure 1: Configurations and absolute positions](image)
Proof. For the first assertion observe that by definition $\xi \in \Omega$ if and only if
$$k_{a,b}(\xi_{i+1}, \xi_i) = 1 \quad \text{for all } i, \text{i.e.},
$$
$$\xi_{i+1} - \xi_i + b = (i+1)b + \xi_{i+1} - ib - \xi_i = ((i+1)b + \xi_{i+1}) - (ib + \xi_i) \geq a.$$

The second assertion is easily verified by induction. The third one is an immediate consequence of the second: Suppose $a > b$ and $\xi = \tau^s \xi \in \Omega \subset \mathbb{F}^\mathbb{N}$. Then $\xi_1 = \xi_{n+1} \geq \xi_1 + n(a - b)$ implied by (ii) gives a contradiction.

Given a configuration $\xi = (\xi_i)_{i \in \mathbb{N}} \in \mathbb{F}^\mathbb{N}$, $p(\xi)_i = \xi_i + ib$ is the position of the (left edge of the) $i$-th rod. Hence $\xi_i$ is the position of the $i$-th rod relative to the lattice point $ib$. Then Prop. 2.1 (i) says that allowed configurations are precisely those for which the rods do not overlap. This justifies our naming hard rod model.

The partition function $Z_n(A(\Delta))$ depends on the values of the observable $A(\Delta)$ on $\tau^n$-periodic sequences. Because of Prop. 2.1 (ii) we assume that $0 < a \leq b$. The shift $\tau: \Omega \to \Omega$, $(\tau \xi)_i = \xi_{i+1}$ on the configuration space induces a shift $\tilde{\tau}: p(\Omega) \to p(\Omega)$, $(\tilde{\tau} \eta)_i = \eta_{i+1} - b$ on the position space such that $p \circ \tau = \tilde{\tau} \circ p$.

Thus $\tau^n$-periodic sequences in configuration space, e.g. $\xi = (\xi_1 \ldots \xi_n) \in \Omega$, correspond to $\tau^n$-periodic sequences in position space. Since $\tilde{\tau}^n \eta = \eta$ holds if and only if $\eta(i+n) = nb + \eta(i)$ for all $i \in \mathbb{N}$, our notion of periodic boundary condition differs from that in [8].

Proposition 2.2. Let $\Delta: \mathbb{R} \to \mathbb{C}$ and $\delta: \mathbb{R} \to [0, \infty]$ be functions such that $|\Delta(x)| \leq \delta(x)$ for all $x \in \mathbb{R}$, $\delta$ is non-increasing, and $\sum_{j \in \mathbb{N}} \delta(aj) < \infty$. The observable $A(\Delta): \Omega \to \mathbb{C}$ (4) associated with $\Delta$ is bounded by a constant which does not depend on $F$.

Proof. Let $\xi \in \Omega$ and $i \in \mathbb{N}$. Then Proposition 2.1 (ii) implies
$$\xi_i - \xi_1 + (i-1)b \geq (i-1)(a - b) + (i-1)b = (i-1)a \geq 0.$$

Now the assumptions on $\Delta$ and $\delta$ show
$$|A(\Delta)(\xi)| \leq \sum_{i=2}^{\infty} |\Delta(\xi_i - \xi_1 + i-1)| \leq \sum_{i=2}^{\infty} \delta(\xi_i - \xi_1 + i-1)b \leq \sum_{i=2}^{\infty} \delta((i-1)a),$$

which is finite.

Examples of hard rod interactions $A(\Delta)$ for which Proposition 2.2 applies are, for instance, $\Delta(x) = \lambda^x p(x)$ for some polynomial $p \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ or $\Delta(x) = (|x| + e)^{-s}$ for some $s > 1$.

3 Transfer operators

In the following we fix a hard rod shift $(F, \nu, A_{a,b}, A(\Delta))$ and consider two special families of distance functions $\Delta$.

Remark 3.1. Let $G$ be a bounded invertible operator on a Hilbert space $\mathcal{H}$. Suppose there exists a cut, i.e., a path $\gamma$ connecting zero with minus infinity, which avoids $\text{spec}(G)$, which, for instance, happens if $G$ has discrete spectrum. Then on $\mathbb{C} \setminus \gamma$ there exists a holomorphic logarithm and hence $G^x$ for $x \in \mathbb{R}$ is
well-defined by the holomorphic functional calculus $G^x = \frac{1}{2\pi i} \oint_\gamma \zeta^x (\zeta - G)^{-1} \, d\zeta$ for any positively oriented rectifiable Jordan contour $\gamma \subset \mathbb{C} \setminus (\gamma \cup \text{spec}(G))$ enclosing $\text{spec}(G)$, cf. [1]. We claim that $\rho_{\text{spec}}(G^x) \leq \rho_{\text{spec}}(G)^x$ for $x > 0$. The spectrum of $G$ is contained in the annulus centered at zero with radii $\rho_1 := \rho_{\text{spec}}(G^{-1})^{-1}$ and $\rho_2 := \rho_{\text{spec}}(G)$. Let $\epsilon > 0$. By deforming the contour $\alpha$ we can assume that $\rho_1 - \epsilon < |z| < \rho_2 + \epsilon$ for all $z \in \alpha$. A standard estimate shows that

$$\|G^x\|^{1/j} \leq \left(\frac{\rho_2 + \epsilon}{2\pi} \oint_\gamma \| (\zeta - G)^{-1} \| \, d\zeta \right)^{1/j} \xrightarrow{\epsilon \to 0} (\rho_2 + \epsilon)^x.$$  

This shows the claim and, similarly, $\rho_{\text{spec}}(G^{-x}) \geq \rho_{\text{spec}}(G^{-1})^{-x}$. In particular, if $\rho_{\text{spec}}(G) < 1$ and $a > 0$, then $\rho_{\text{spec}}(G^a) \leq \rho_{\text{spec}}(G)^a \leq 1$. Moreover, $\rho_{\text{spec}}(G^a)$ and, by a similar argument, $\|G^x\|$ tend to zero as $x$ tends to infinity.

**Lemma 3.2.** (A) Let $G$ be a bounded invertible operator on a Hilbert space $\mathcal{H}$ with spectral radius less than one and discrete spectrum. Let $v, w \in \mathcal{H}$ and define $\Delta : \mathbb{R} \to \mathbb{C}$, $\Delta(x) := (G^x v, w)$. We set $G_x := G^x$.

(B) Let $F \subset [0,b]$ and $(G_t)_{t \geq 0}$ be a $C_0$-semigroup of operators on a Hilbert space $\mathcal{H}$ with $\rho_{\text{spec}}(G_0), \rho_{\text{spec}}(G_t) < 1$. Let $v, w \in \mathcal{H}$ and define $\Delta : [0, \infty) \to \mathbb{C}$, $\Delta(x) := (G_x v, w)$.

In both cases the defining series of the observable converges absolutely and can be represented as

$$A_\Delta(\xi) = \sum_{j=2}^{\infty} \langle G_{j-2}^x G_{\xi_j} v | (G_{b-\xi_1})^x w \rangle.$$

**Proof.** In both cases the function $\Delta$ is dominated by $\delta(x) := \|G_x\| \|v\| \|w\|$. We claim that $\delta$ satisfies the summability condition $\sum_{j=1}^{\infty} \delta(a_j) < \infty$ from Proposition 2.2. Then for any $\xi \in \Omega$ one has

$$A_\Delta(\xi) = \sum_{j=2}^{\infty} \langle G_{\xi_j + j} G_{b-\xi_1} v | w \rangle = \sum_{j=2}^{\infty} \langle G_{b}^{j-2} G_{\xi_j} v | (G_{b-\xi_1})^x w \rangle.$$

It suffices to show that $\lim_{j \to \infty} \|G_{a_j}\|^{1/j} = \rho_{\text{spec}}(G_a) < 1$. In case (B) this holds by assumption. In case (A) we use Remark 3.1.

Lemma 3.2 is the key step in our construction of the transfer operator for the HRM. We apply the methods from [5, Sec. 5] setting

$$G := G_b, \quad a_x := \frac{\gamma}{\pi} (G_{b-x})^x w, \quad b_x := G_x v, \quad \text{and } q_x := 0.$$

Then [5, Thm. 5.3] shows that the iterates $\mathcal{M}_\beta^n$ of the Ruelle-Mayer type operator

$$\langle \mathcal{M}_\beta f, \sigma, z \rangle = \int_F A_{a,b}(x, \sigma) \exp(\beta |z| \langle (G_{b-x})^x w \rangle) f(x, G_x v + G_b z) \, d\nu(x)$$

$^1 \mathcal{F}(\mathcal{H})$ denotes the Fock space over $\mathcal{H}$, i.e., the unique Hilbert space with reproducing kernel $\exp(\pi(x|y)_{\mathcal{H}})$, cf. [2], [9], or [5, Sec. 3].
are eventually of trace class on $L^2(F, dv) \otimes \mathcal{F}(\mathcal{H})$ under the following two conditions: First, the operator $G_b$ belongs to some Schatten class $S_p(\mathcal{H})$ for $p < \infty$, defined for instance in [4]. For interactions of type (A) this, together with the invertibility of $G$ required in Lemma 3.2, implies that $\mathcal{H}$ is necessarily finite dimensional. Secondly, an integrability condition holds true, which depends on the following data: For all $n \geq n_0$ such that $\|G_b^n\| < 1$ and $x = (x_1, \ldots, x_n) \in F^n$ we set - according to [5, equations (24), (25)] -

\[ q(n; x) := \pi \sum_{k=1}^{n} \sum_{j=1}^{n-k} (G_b^{j-1} b_{x+j+k} (a_{x_k}) = \beta \sum_{k=1}^{n} \sum_{j=1}^{n-k} (G_{(j+b)x+k} v|w), \]

\[ a(n; x) := \sum_{k=1}^{n} (G_b^{n-k})^* a_{x_k} = \beta \sum_{k=1}^{n} (G_{(n-k+1)b-x_k})^* w, \]

\[ b(n; x) := \sum_{j=0}^{n-1} G_{j+b} x_{j+1} = \sum_{j=0}^{n-1} G_{j+b} x_{j+1} v. \]

For sufficiently large $n$ we define the associated function $c(n; \cdot) : F^n \rightarrow \mathbb{C}$,

\[ (7) \quad c(n; x) := \exp \left( 2\text{Re}(q(n;x)) + \pi \|a(n;x)\|^2 + \pi \| (1-G_b^n(G_b^n)^*)^{-1/2} G_b^n (a(n;x) + b(n;x)) \|^2 \right). \]

**Theorem 3.3.** Let $(F, \nu, \alpha_{a,b}, A(\Delta))$ be a hard rod subshift associated with $\Delta : \mathbb{R} \rightarrow \mathbb{C}$, $\Delta(x) := (G_x v|w)$ belonging to (A) or (B) from Lemma 3.2. Suppose there exists $n \in \mathbb{N}$ such that $\int_{F^n} c(n; x) dv^n(x) < \infty$.

(i) Then there exists an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the iterates $\mathcal{M}_\beta \in \text{End}(L^2(F, dv) \otimes \mathcal{F}(\mathcal{H}))$ of the Ruelle-Mayer type operator $\mathcal{M}_\beta$ (6) are of trace class and satisfy the dynamical trace formula

\[ Z_n(\beta A(\Delta)) = \det(1-G_b^n) \text{ trace } \mathcal{M}_\beta. \]

(ii) The dynamical zeta function $\zeta_R(z; \beta) := \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta A(\Delta)) \right)$ has a meromorphic continuation to $(z, \beta) \in \mathbb{C}^2$ via

\[ \zeta_R(z; \beta) = \exp \left( \sum_{n=1}^{n_0-1} \frac{z^n}{n} Z_n(\beta A(\Delta)) \right) \prod_{j=0}^{\dim \mathcal{H}} \left( \det_{n_0} (1-z \mathcal{M}_\beta \otimes \wedge^j G_b) \right)^{(-1)^{j+1}}. \]

**Proof.** Part (i) is a consequence of [5, Thm. 5.3]. For (ii) we invoke [5, Cor. 2.2] yielding the meromorphic continuation to the entire complex $z$-plane. By [5, Prop. 1.5. (ii)] one has the expansion

\[ \prod_{j=0}^{\infty} \det_{n_0} (1-z \mathcal{M}_\beta \otimes \wedge^j G_b) = \lim_{d \rightarrow \infty} \prod_{\alpha \in \{0, 1\}^d} \det_{n_0} (1-z \lambda^\alpha \mathcal{M}_\beta), \]

where the $\lambda_j$’s are the eigenvalues of $G_b$. The operator $\mathcal{M}_\beta$ and thus the determinant $\det_{n_0} (1-z \mathcal{M}_\beta \otimes \wedge^j G_b)$ depend holomorphically on $\beta$ for all fixed $z \in \mathbb{C}$ and $j \in \mathbb{N}_0$. Since $\sum_{\alpha \in \{0, 1\}^d} \| (z \lambda^\alpha \mathcal{M}_\beta)^{n_0} \|$ converges locally uniformly in $\beta$ as $d \rightarrow \infty$, one shows along the lines of [5, Lemma 1.3] that the sequence of holomorphic functions \[ f_d : \beta \mapsto \prod_{\alpha \in \{0, 1\}^d} \det_{n_0} (1-z \lambda^\alpha \mathcal{M}_\beta) \]

converges locally uniformly as $d \rightarrow \infty$ and hence zeta is a meromorphic function in $\beta \in \mathbb{C}$, too. \[ \square \]
In the following we investigate the two families of hard rod models introduced
in Lemma 3.2. We begin with type (A). Using Remark 3.1 one confirms that the
sum \( \|a_z\| + \|b_z\| \) defined in (5) is of order \( \rho_{\text{spec}}(G_0)^{-|x|} \) as \( |x| \to \infty \). Hence \( c(n, \cdot) \) grows double-exponentially, i.e., like \( \exp(\exp(C|x_i|)) \) with \( C > 0 \), in each
direction \( |x_i| \to \infty \). In order to apply [5, Thm. 5.3] we have to require that
\( \int_{F_n} c(u; x) \, dv^\alpha(x) < \infty \) for all \( n \geq n_0 \), i.e., the a priori measure \( \nu \) on \( F \) has a
strong decay at infinity. In the following we will characterize the class of
distance functions.

**Proposition 3.4.** The following are equivalent for a function \( \Delta : \mathbb{R} \to \mathbb{C} \).

(i) There exist \( \lambda_k \in \mathbb{D}^\infty := \{ z \in \mathbb{C}; 0 < |z| < 1 \} \) and polynomials \( p_k \in \mathbb{C}[x] \)
\((k = 1, \ldots, n)\) such that \( \Delta(x) = \sum_{k=1}^n p_k(x) \lambda_k^x \).

(ii) There exists a finite dimensional Hilbert space \( \mathcal{H} \), an invertible operator
\( G \) on \( \mathcal{H} \), and vectors \( v, w \in \mathcal{H} \) such that \( \Delta(x) = (G^x v|w) \).

The proof of Proposition 3.4 is based on the following Lemma.

**Lemma 3.5.** Let \( p \in \mathbb{N}_0 \) and \( B^{(p)} \in \text{Mat}(p+1, p+1; \mathbb{R}) \) be the unipotent (lower)
triangular matrix with entries

\[
(B^{(p)})_{i,j} = \begin{cases} \binom{i}{j}, & \text{if } 0 \leq j \leq i \leq p, \\ 0, & \text{otherwise.} \end{cases}
\]

Then for all \( z \in \mathbb{C} \) the matrix \((B^{(p)})^z\) defined via holomorphic functional calculus
is given by

\[
((B^{(p)})^z)_{i,j} = \begin{cases} z^{i-j} \binom{i}{j}, & \text{if } 0 \leq j \leq i \leq p, \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** Near \( u = 1 \) the function \( f_z(u) = u^z \) is holomorphic and its derivatives
are \( \partial_u^i f_z(u) = \prod_{k=0}^{i-1} (z - k) u^{z-i} \). The holomorphic functional calculus (cf. [1,
VII.1]) shows that for any unipotent matrix \( B \in \text{Gl}(p+1; \mathbb{C}) \) one has

\[
B^z = \sum_{i=0}^p \frac{\partial_u^i f_z(1)}{i!} (B - 1)^i = 1 + \sum_{i=1}^p \prod_{k=0}^{i-1} (z - k) \frac{(B - 1)^i}{i!}.
\]

Hence for any \( 0 \leq i, j \leq p \) the coefficients \((B^z)_{i,j}\) polynomially depend on \( z \)
(and polynomially on \( B \), too). Thus it suffices to verify that (8) holds for all
\( z \in \mathbb{N}_0 \). For \( i \geq j \) one has

\[
((B^{(p)})^z)_{i,j} = \sum_{k=j}^i n^{i-k} \binom{i}{k} \binom{k}{j} = \binom{i}{j} \sum_{l=0}^{i-j} n^{i-j-l} \binom{i-j}{l} = \binom{i}{j} (n+1)^{i-j}
\]

and hence induction concludes the proof. \( \square \)

**Proof of Prop. 3.4.** Given \( \lambda_k \in \mathbb{D}^\infty \) and polynomials \( p_k \in \mathbb{C}[x] \) \((k = 1, \ldots, n)\),
one chooses a cut. This determines a branch of \( f_z(u) = u^z \) and Lemma 3.5 shows that
the function \( \Delta(x) := \sum_{k=1}^n p_k(x) \lambda_k^x \) can be represented as \( \Delta(x) = (G^x v|w) \)
where \( G = \bigoplus_{k=1}^n \lambda_k B^{(\deg p_k)} \) acting on \( \mathcal{H} := \mathbb{C}^{n,\Delta} \) for \( n_\Delta := n + \sum_{k=1}^n \deg p_k \).
Conversely, we decompose any given invertible operator $G$ on a finite dimensional Hilbert space with spectral radius less than one into its Jordan blocks. We note that $\lambda S^{(p)}$ is conjugate to the Jordan block of size $p + 1$ and eigenvalue $\lambda$ and observe that $(SGS^{-1})^x = SG^xS^{-1}$. Hence every representation $\Delta(x) = \langle G^xv|w \rangle$ is of the form $\Delta(x) = \sum_{k=1}^n p_k(x) \lambda_k^x$ as above.

We end with hard rod shifts with interaction of type (B).

Remark 3.6. (i) We say $\xi \in \Omega$ contains a cluster of $n$ rods, if there exists $i_0 \in \mathbb{N}$ such that $\xi_{i_0-1} + a - b < \xi_{i_0}$, $\xi_{i_0} + (n - 1)(a - b) = \xi_{i_0+n-1}$, and $\xi_{i_0+n-1} + a - b < \xi_{i_0+n}$. Using the position map $p : F^N \rightarrow \mathbb{R}^N, (\xi_i)_{i \in \mathbb{N}} \mapsto (\xi_i + ib)_{i \in \mathbb{N}}$ this is equivalent with $p(\xi)_{i_0-1} + a < p(\xi)_{i_0}$, $p(\xi)_{i_0} + (n - 1)a = p(\xi)_{i_0+n-1}$, and $p(\xi)_{i_0+n-1} + a < p(\xi)_{i_0+n}$, i.e., the $n$ rods $i_0, \ldots, i_0 + n - 1$ lie side by side.

(ii) If $F \subset [0, b]$, then only clusters of length $n \leq 1 + b(b-a)^{-1}$ can occur, since $(n-1)(b-a) = \xi_{i_0} - \xi_{i_0+n-1} \in [-b, b]$. This suggests the naming mock hard rod model for the hard rod shift $(F, \nu, \mathcal{A}(\Delta))$ with observables $A(\Delta)$ of type (B).

Example 3.7. (i) Let $\lambda \in \ell^1\mathbb{N}$ with $\|\lambda\|_{\ell^1\mathbb{N}} < 1$ and $c \in \ell^1\mathbb{N}$. Choosing (if necessary) for each $\lambda_j$ a branch of log, we define a operator valued function $[0, \infty) \rightarrow \text{End}(\ell^2\mathbb{N}), x \mapsto G_x := \text{diag}(\lambda_k^x)_{k \in \mathbb{N}}$. Clearly, $(G_x)_{x \geq 0}$ is a $C_0$-semigroup with $G_t \in S_p(\ell^2\mathbb{N})$ where $p = \max(1, q/b)$. Then $\Delta(x) := \sum_{k=1}^\infty \xi_k \lambda_k^x$ is of the form $\Delta(x) = \langle G_xv|w \rangle$ where $v = \sqrt{c} \in \ell^2\mathbb{N}$.

(ii) Let $f \in \mathcal{O}(\mathbb{D})$ be a holomorphic function on the unit disk $\mathbb{D}$ with $f(0) = 0$ such that $\sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!}$ converges absolutely. Then for any $\lambda \in \mathbb{D}$ (and a branch of log) the function $\Delta(x) := f(\lambda^x) = \sum_{k=1}^\infty \frac{f^{(k)}(0)}{k!} \lambda^{kx}$ is of type (i).

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References


