ON THE ALGEBRAIC COMPLEXITY OF SOME FAMILIES OF COLOURED TUTTE POLYNOMIALS

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ABSTRACT. We investigate the coloured Tutte polynomial in Valiant's algebraic framework of **NP**-completeness. Generalising the well known relationship between the Tutte polynomial and the partition function from the Ising model, we establish a reduction from the permanent to the coloured Tutte polynomial, thus showing that its evaluation is a **VNP**−COMPLETE problem.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. The Turing complexity of the Tutte polynomial.

The Tutte polynomial plays a central role in the study of graph invariants, as testified by a whole chapter dedicated to it in the recent, and extremely enjoyable, monograph by B. Bollobás, [Bol98]. It has been the object of complexity investigations ever since the influential work of Jaeger, Vertigan and Welsh [JVW90]. They study the worst case complexity in the framework of Turing machines. It is shown that the problem of evaluating the Tutte polynomial on points (a, b) from a finite algebraic extension of $\mathbb{Q}$ is **#P**−HARD, provided these points do not belong to some finite set of exceptions. Computing all the coefficients of the Tutte polynomial is **#P** − HARD under Turing reductions, since any algorithm doing so can be used to get the number of 3-colourings of a graph (a known **#P**−COMPLETE problem). It was shown by J.D. Annan [Ann95] that computing a coefficient of the Tutte polynomial is in **#P**, and that some of the coefficients can be computed in polynomial time, while others are **#P**−COMPLETE. Evaluating the Tutte polynomial at a point is not in **#P** (since it can evaluate to negative values) but is in **GapP**, the closure **#P** under subtraction. For more on the class **#P** see for example C. Papadimitriou's monograph [Pap94, chapter 18], L. Valiant's original work [Val79b], or the delightful book by D. Welsh, [Wel93b]. The class **GapP** was introduced in [FFK94], see [For97] for a nice survey. The Fixed Parameter Tractability in the sense of [DF99] of the Tutte polynomial is studied in [OW92, And98, Nob98, Nob97, Mak01]. It is shown that the Tutte polynomial and most of its relatives are polynomial time computable on series-parallel graphs, and more generally, on graphs of bounded treewidth. Its approximation complexity is studied in [Wel93a, AFW94, AFW95, Wel95, Wel97, Wel99, Kar99]. It is shown that fully polynomial randomized approximation schemes (FPRAS) do not exist unless **RP** = **NP**, but they do exist for dense graphs and for graphs with no small edge cut set. A full classification of approximability of the Tutte polynomial is still not known.

1.2. **Algebraic complexity.** The Tutte polynomial is, despite its combinatorial description, inherently an algebraic object. This leads to the problem of finding the right algebraic computational model in which to study its complexity. A model suited for algebraic computation is Valiant's non-uniform model of computation [Val79a]. The complexity of a multivariate polynomial in this model is basically the minimal amount of arithmetic operations needed to build this polynomial from its indeterminates and constants. The analogies to **P** and **NP** in this model are the classes **VP** and **VNP** of families of polynomials. A family (f_n) of polynomials is *reducible* to a a family (g_n) of polynomials *via p-projections*, if there is a polynomially bounded function $p\colon \mathbb{N} \to \mathbb{N}$ such that

$$f_n(X_1, \ldots, X_n) = g_{p(n)}(a_1, \ldots, a_{p(n)}),$$

where the a_i are constants or variables among $\{X_1, \ldots, X_n\}$.

A family (g_n) of polynomials in **VNP** is *complete* via p-projections, if all families $(f_n) \in$ **VNP** are p-projections of (g_n). p-projections are transitive, so in order to show that a family of polynomials is **VNP** − COMPLETE it suffices to show that it projects to a known family of complete polynomials. The classical complete problem in **VNP** is the permanent family

$$PER_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)},$$

where S_n is the symmetric group. Typical families of functions in **VNP** are generating functions of graph properties. These are polynomials depending on graphs with indeterminate weights X_e on the edges. The complexity of generating functions of graph properties in this model was first studied by Jerrum [Jer81] and further developed by Bürgisser [Bür00a]. One such family is the family of generating functions of closed subgraphs of a graph $G = (V, E)$:

$$C_G(X_e; e \in E) = GF(G, \mathcal{IS}) = \sum_{\substack{E' \subseteq E \\ (V,E') \in \mathcal{IS}}} \prod_{e \in E'} X_e,$$

where $\mathcal{IS}$ denotes the class of closed graphs, i.e. graphs in which all vertices have even degree, possibly zero. For cubic lattice graphs, this family was shown to be **VNP** − COMPLETE by Jerrum [Jer81] (see Proposition 7 in section 2.1 for the definition of a cubic lattice graph).

Note that the Tutte polynomial cannot be a complete problem in **VNP**, since it depends on only two variables. In order to get a complete family of polynomials, we would have to put weights on the edges of the underlying graphs. This leads to a weighted Tutte polynomial that is a variation of Traldi's dichromatic polynomial [Tra89]. But we shall first pursue a different line of investigation.

1.3. **Coloured Tutte polynomials.** The object of our investigation will be the Tutte polynomial for coloured graphs as introduced by Bollobás and Riordan in [BR99]. The coloured Tutte polynomial $W_{G,c,\phi}$ depends on a graph $G = (V, E)$ with an edge-colouring $c\colon E \to \Lambda$ and an ordering ϕ on the edges. It is built from four indeterminates $X_\lambda, Y_\lambda, x_\lambda, y_\lambda$ for each colour

$\lambda \in \Lambda$. Since the coloured Tutte polynomial is a function on coloured, ordered graphs and Valiant's complexity theory deals with families of polynomials, our first task is to construct suitable families of polynomials. We will show in Theorem 10 that there are families of graphs for which the corresponding family of coloured Tutte polynomials lies in **VNP**. Since the classical Tutte polynomial is a substitution of the coloured Tutte polynomial, we can also get families of classical Tutte polynomials in **VNP**. To show that a family of coloured Tutte polynomials is **VNP**−COMPLETE, we will construct a family of graphs so that their coloured Tutte polynomials project to the generating function of closed subgraphs for cubic lattice graphs.

1.4. **Main results.** Our first theorem shows that a substitution instance of the coloured Tutte polynomial gives the generating function of closed subgraphs. It is basically a consequence of some well known relationships between the Tutte polynomial and generating functions arising in the context of statistical mechanics.

Theorem 1. *Let $G = (V, E)$ be an edge-ordered, maximal coloured graph, and $C_G(w_e) = GF(G, \mathcal{IS})$. If for every edge $e \in E$ we have*

$$X_{c(e)} = 2, \quad Y_{c(e)} = 1 + w_e, \quad x_{c(e)} = 2w_e, \quad y_{c(e)} = 1 - w_e,$$

then the following identity holds:

$$C_G(w_e) = W_{G,c}(X_{c(e)}, Y_{c(e)}, x_{c(e)}, y_{c(e)}).$$

Here, maximal coloured means that each edge maps to a different colour. A modification of the graphs will then give us the following theorem.

Theorem 2. *There is a family of edge-ordered, maximal coloured graphs (G_n, c_n), such that the corresponding family of coloured Tutte polynomials W_{G_n,c_n} is **VNP**−COMPLETE under p-projections.*

The proof is given in section 3. In the proof we will have to keep an eye on the difficulties arising from the fact that the coloured Tutte polynomial is not independent of the order of the edges.

An immediate consequence is the following corollary.

Corollary 3. *Let M be a $(n \times n)$-matrix with indeterminate entries. Then there is a maximal coloured graph G, with number of edges polynomially bounded in n, so that the permanent $per(M)$ of M can be obtained from the coloured Tutte polynomial W_G by substitution of its indeterminates with entries from M or constants.*

The corollary obviously holds also when replacing the permanent with the determinant or the hamiltonian, or any other family of polynomials in **VNP**. Conversely, as the permanents PER_n of $(n \times n)$-matrices form an **VNP**−COMPLETE family, the converse is also true, i.e. all the variations of the Tutte polynomial can be obtained from PER_n, for suitably chosen n, by substitution instances.

The permanent of a $(0, 1)$-matrix can be viewed as counting the perfect matchings of the underlying bipartite graph. There has been previous work relating the Tutte polynomial to matching. In [NW99], the authors consider a generalisation of the Tutte polynomials for weighted graphs which specialises to the matching polynomial. A relation between the Tutte polynomial and the matching polynomial also arises from Stanley's symmetric function generalisation of the chromatic polynomial [Sta95, Sta98].

1.5. **The weighted Tutte polynomial.** We have seen above that the classical Tutte polynomial cannot be a complete problem in **VNP**, since it depends on only two variables. In order to remedy this, we put weights on the edges of the underlying graphs and obtain a weighted Tutte polynomial that is a variation of Traldi's dichromatic polynomial [Tra89].

Definition 1. (dichromatic polynomial) For a graph $G = (V, E)$, the polynomial

$$Q(G; v_e, q, z) = \sum_{E' \subseteq E} \Big(\prod_{e \in E'} v_e \Big) q^{k\langle E' \rangle} z^{|E'| - r\langle E' \rangle}$$

is called the *dichromatic polynomial* of G.

See subsection 2.2 for the definition of $k\langle E' \rangle$ and $r\langle E' \rangle$. By the relationship to the generating function of closed subgraphs, Lemma 15 in section 3, we get immediately for oracle reductions:

Theorem 4. *$Q(G; v_e, q, z)$ is **VNP**−* COMPLETE *via polynomial oracle reductions.*

Oracle reductions are a more liberal notion of reduction than p-projections. They are similar to Turing reducibility, and will be defined in section 2.1. It is not clear whether this is also true for p-projections.

Problem 1. Is Traldi's dichromatic polynomial $Q(G; v_e, q, z)$ complete in **VNP** via p-projections?

1.6. **Outline of the paper.** We start in section 2 by recalling some of the theory underlying our result. In subsection 2.1 we review Valiant's model of computation of algebraic straight-line programs, and sketch its complexity theory according to Valiant. In subsection 2.2 we present the basics about the Tutte polynomial for coloured graphs. In subsection 2.3 we show how the coloured Tutte polynomials fit into Valiant's framework, and in subsection 2.4 we recall the connection between the Tutte polynomial and statistical mechanics which motivated our generalisation, Lemma 15 in section 3. Section 3 contains the proofs of Theorems 1 and 2. For the proof of Theorem 1 we need four technical lemmas, subsection 3.1. For the proof of Theorem 2 we need a generalisation, Lemma 17, of a theorem from [BR99], given in subsection 3.2. The proof is then completed in subsection 3.3. In section 4, finally, we draw conclusions and discuss further research.

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2. Background material

2.1. **Non-uniform algebraic NP-completeness.**

An algebraic framework of polynomial time computability with an analogue of **NP**-completeness was introduced by Valiant [Val79a], in order to study the computational problem of evaluating multivariate polynomials. One of the first results was a formal analysis of why the permanent seems hard to compute, while the determinant is easy. Comprehensive accounts on this theory can be found in [BCS97, Chapter 21] and, with somewhat more detail, in [Bür00a].

In what follows, a polynomial always means a multivariate polynomial over some field $\mathbb{F}$.

Definition 2. (Straight-line program) A *straight-line program* Γ of *size* r, with input size m, consists of a sequence of instructions $(\Gamma_1, \ldots, \Gamma_r)$, where each instruction Γ_k consists of an operation symbol $\omega_k \in \{+, -, *\}$ and two addresses (pointers) i_k and j_k, so that $-m < i_k, j_k < k$.
The *result sequence* of a straight-line program Γ on a sequence of input polynomials $a_1, \ldots, a_m$ is the unique sequence

$$(b_{-m+1}, \ldots, b_r),$$

where $b_k = a_{m+k}$ for $k \leq 0$ and $b_k = b_{i_k} \omega_k b_{j_k}$ else.
A straight-line program is said to *compute* a set of polynomials F from an input sequence $a_1, \ldots, a_m$, if the polynomials in F are among the result sequence.

A straight-line program can be viewed as an acyclic, directed graph with m input nodes labeled $-m+1, \ldots, 0$, r computation nodes labeled $1, \ldots, r$, and arrows joining nodes i_k and j_k to node k, for $1 \leq k \leq r$. In most cases, the input sequence will consist of indeterminates X_i and constants from some field $\mathbb{F}$. The complexity of a multivariate polynomial is the minimal amount of arithmetic operations in such an algorithm.

Definition 3. (Complexity) The *complexity* $L(f)$ of some polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ is the minimal size of a straight-line program computing f from inputs among the $X_i, 1 \leq i \leq n$, and constants from the field $\mathbb{F}$.

We will be dealing with families of polynomials rather than single polynomials. For convenience, a family $(f_n)_{n\in\mathbb{N}}$ will simply be denoted by (f_n). The fact that we do not assume a "machine" dealing with such a family, and each polynomial in the family has to be handled by a separate straight-line program, accounts for the term *non-uniform.*

A function $t : \mathbb{N} \to \mathbb{N}$ is called *p-bounded (from above)*, if there is a $c \geq 0$ such that for all $n \in \mathbb{N}$, $t(n) \leq n^c + c$. A family of polynomials (f_n) is called a *p-family*, if the degree and the number of variables are p-bounded functions in n.

Definition 4. (p-computable) A p-family (f_n) is said to be *p-computable*, if the complexity of f_n is p-bounded in n. The class of p-computable families of polynomials is denoted by **VP**.

Note that **VP** depends on the field $\mathbb{F}$ under consideration. A basic example of a family of polynomials in **VP** is the determinant family of matrices with entries $X_{ij}, 1 \leq i, j \leq n$. Gauss elimination yields an evaluation that is polynomially bounded in n. We note that even though Gauss elimination involves divisions, these can be avoided with only a polynomial increase in complexity due to a result of V. Strassen [Str73]. In analogy to the class **P** in the Turing machine model, **VP** consists of families of polynomials considered to be tractable. The analogy to **NP** is the class **VNP** of p-definable families of polynomials. To be precise, **VNP** resembles more the class **#P**, see [Bür00b] for details, but its role within the theory is similar to that of **NP** in Turing complexity.

Definition 5. (p-definable) A p-family (f_n) is said to be *p-definable*, if there is a p-computable family (g_n) such that for every n, there is a polynomial q, such that

$$f_n(X_1, \ldots, X_{p(n)}) = \sum_{e \in \{0,1\}^{q(n)}} g_n(X_1, \ldots, X_{p(n)}, e_1, \ldots, e_{q(n)}).$$

The class of p-definable families of polynomials is denoted by **VNP**.

Obviously, $\mathbf{VP} \subseteq \mathbf{VNP}$. As in classical complexity theory, a major open problem is whether every p-definable family is p-computable. An example of a p-definable family, not assumed to be p-computable over fields of characteristic $\neq 2$, is the permanent family

$$PER_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}.$$

The following is useful for proving p-definability. It is called in [Bür00a] *Valiant's criterion.*

Proposition 5. *(Valiant's criterion) If $\phi : \{0,1\}^* \to \mathbb{N}$ is a function in #**P**, then the family*

$$f_n = \sum_{e \in \{0,1\}^n} \phi(e) X_1^{e_1} \cdots X_n^{e_n}$$

is p-definable.

In order to compare the degrees of difficulty of problems, a notion of reducibility is needed. The classical notion in algebraic complexity is that of a p-projection.

Definition 6. (projection) Let $f \in \mathbb{F}[X_1, \ldots, X_n]$ and $g \in \mathbb{F}[Y_1, \ldots, Y_m]$, with $n \leq m$. Then, f is called a *projection* of g, if f can be written as

$$f(X_1, \ldots, X_n) = g(a_1, \ldots, a_m),$$

for $a_i \in \mathbb{F} \cup \{X_1, \ldots, X_n\}$, $1 \leq i \leq m$.

Definition 7. (p-projection) A p-family $f = (f_n)$ is called a *p-projection* of a family $g = (g_n)$, $(f \preceq_p g)$, if there is a p-bounded function $t \colon \mathbb{N} \to \mathbb{N}$, such that for all $n \in \mathbb{N}$, f_n is a projection of $g_{t(n)}$.

It can easily be checked that the p-projection is transitive and **VP** as well as **VNP** are closed under p-projections. The p-projection is rather weak, since it does not fully grasp the intuitive concept *evaluating g is a least as hard as evaluating f*. It does not even serve well as an analogy to many-one reductions in the Turing model. Instead, the projection represents the concept that f can be computed by simply running g on a different set of inputs. For a discussion, why p-reductions were chosen as the basic notion of reduction, cf. [SV85]. There, the authors consider projections of boolean functions and argue that this is essentially sufficient to describe most of the reductions used in Turing complexity theory.

A more liberal notion of reduction is that of a c-reduction, or *polynomial oracle* reduction, as introduced in [Bür00a, Chapter 5]. It is analogous to the concept of a Turing reduction. Call a straight-line program with oracle polynomial g a program, which in addition to the arithmetic operations $\{+, *, -\}$, has an operation for evaluating g from a given set of intermediate results.

Definition 8. (Oracle complexity) The *oracle complexity* of a polynomial f with respect to a polynomial g, denoted $L^g(f)$, is the minimal size of an straight-line program with oracle g computing f from a set of indeterminates and constants.

Definition 9. (c-reduction) A p-family $f = (f_n)$ is called a *c-reduction*, or *polynomial oracle reduction*, of a p-family $g = (g_n)$, $(f \preceq_c g)$, if there is a p-bounded function $t : \mathbb{N} \to \mathbb{N}$ such that the oracle complexity $L^{g_{t(n)}}(f_n)$ is a p-bounded function in n.

Having discussed notions of reductions, we can now introduce the concept of completeness in our context.

Definition 10. (**VNP** − COMPLETE) A p-family $f \in$ **VNP** is said to be **VNP**−COMPLETE with respect to a reduction $\preceq$, if $g \preceq f$ for any family g in **VNP**.

Transitivity implies that, if g is **VNP**−COMPLETE and $g \preceq f$, then also f is **VNP**−COMPLETE. The classical example of a **VNP**−COMPLETE problem via p-projections is the permanent family for fields of characteristic $\neq 2$.

Proposition 6. *(Valiant,* [Val79a, Val82]*) The permanent family is* ***VNP***– COMPLETE *with respect to p-projections over fields of characteristic* $\neq 2$.

Many p-definable families of polynomials arise from weighted graphs.

Example 1. Let $K_{n,n} = (V = \{1, \dots, n\} \dot\cup \{1, \dots, n\}, E)$ be the complete bipartite graph with $2n$ vertices and edge-weights X_{ij} for an edge joining i and j. Let M be the adjacency matrix of $K_{n,n}$. Then, the permanent of M is given by

$$PER_n = per(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma(j)} = \sum_{E' \subseteq E} \prod_{(i,j) \in E'} X_{ij}$$

where the second sum is over the edge sets of perfect matchings of $K_{n,n}$.

More generally let $\mathcal{E}$ be a graph property, i.e. a class of graphs closed under isomorphism. Let $G = (V, E)$ be an edge-weighted graph with weights X_e. Then the *generating function* of G with respect to $\mathcal{E}$ is defined as:

$$GF(G, \mathcal{E}) = \sum_{\substack{E' \subseteq E \\ (V,E') \in \mathcal{E}}} \prod_{e \in E'} X_e.$$

With this notation, the permanent family can be rewritten as

$$PER = (PER_n) = (GF(K_{n,n}, \mathcal{DI})),$$

where $\mathcal{DI}$ is the class of graphs in which every connected component has exactly two vertices (dimer coverings). Another generating function, which will be of interest later, is given by the property $\mathcal{IS}$ of closed graphs (graphs in which every vertex has even degree, possibly zero). For an edge-weighted graph $G = (V, E)$ define $C_G(X_e; e \in E) = GF(G, \mathcal{IS})$.

Call C_n the cubic lattice graph, with vertex set $\{(i, j, l) \in \mathbb{Z}^3; 1 \le i, j \le n, 0 \le l \le 1\}$ and edges joining the vertices separated by unit distance. The following result is due to Jerrum [Jer81].

Proposition 7. *(Jerrum) The family of generating functions*

$$(GF(C_n, \mathcal{IS}))$$

is ***VNP***$-$COMPLETE *via p-projections.*

This result will later be used to show that there is a **VNP**$-$COMPLETE family of coloured Tutte polynomials with respect to p-projections. More details on generating functions of graph properties in the context of algebraic complexity theory can be found in [Bür00a, Chapter 3].

2.2. **The coloured Tutte polynomial.** The Tutte polynomial for coloured graphs was introduced by Bollobás and Riordan in [BR99]. We follow closely their exposition from which we repeat here what is needed for our investigation.

In what follows, a graph will be allowed to have multiple edges and loops. A graph without multiple edges or loops will be called *simple*. Let $G = (V, E)$ be a graph. Given an edge e, the *deletion* of e from G denotes the graph $G - e := (V, E \backslash \{e\})$. The *contraction* of G at e is the graph G/e, obtained after removing e and pasting the end-vertices together. The number of connected components of a graph G will be denoted by $k(G)$. For an edge set $E' \subseteq E$, denote by $k\langle E'\rangle$ the number of components of the spanning subgraph (V, E'). The *rank* of an edge set $E' \subseteq E$ is given by $r\langle E'\rangle = |V| - k\langle E'\rangle$. The rank of a graph $G = (V, E)$ is then defined as $r(G) = r\langle E\rangle$. A *bridge* in a graph G is an edge e, such that $G - e$ has one more connected component than G. Let G be a connected graph with an ordering of the edges and $T \subseteq E$ a spanning tree. An edge $e \in T$ is called internally active if it is the first edge (with respect to the ordering) in the cut it defines, otherwise it is called internally inactive. An edge $e \in E \backslash T$ is called externally active, if it is the first edge in the unique cycle of the graph induced by $T \cup \{e\}$, otherwise it is called externally inactive. Moreover, a subgraph is called a spanning forest, if it defines a spanning tree on each connected component.

In this work, several polynomials defined on graphs will be considered. These will appear in two different contexts: as *maps* from some class of graphs $\mathcal{G}$ into a polynomial ring, or as *images* of a certain graph G under this map, i.e. as fixed polynomials. Let

$$f : \mathcal{G} \to \mathbb{Z}[\bar{x}]$$

be such a map. Write $f(G; \bar{x})$, or simply $f(G)$, if the focus is on f being a function and $f_G(\bar{x})$ for the polynomial corresponding to a fixed G. In the second case, G will mostly be assumed to be connected and simple. Sometimes the map f will go into a polynomial ring with countably many indeterminates x_i, $i \in \mathbb{N}$. In this case, if the image of a graph G is a polynomial in indeterminates $x_{i_1}, \ldots, x_{i_n}$, consider f_G to be a polynomial in the subring $\mathbb{Z}[x_{i_1}, \ldots, x_{i_n}]$. Moreover, if the indeterminates depend on the edges of the graph (for example if $f_G = f_G(x_e; e \in E(G))$), then simply write $f_G(x_e)$.

We start by introducing the Tutte polynomial for coloured graphs, following [BR99]. Let $G = (V, E)$ be a graph with a colouring $c : E \to \Lambda$ of the edges and an edge ordering $\phi : E \to \{1, \ldots, |E|\}$. Assign to each colour $\lambda \in \Lambda$ four variables $X_\lambda, Y_\lambda, x_\lambda, y_\lambda$ and set $\mathbb{Z}_\Lambda = \mathbb{Z}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda; \lambda \in \Lambda]$. The coloured Tutte polynomial of a connected graph G is the polynomial in $\mathbb{Z}_\Lambda$ defined by:

$$W(G, c, \phi) = \sum_{T \subseteq E} \prod_{\substack{e \in T \\ int.active}} X_{c(e)} \prod_{\substack{e \in E \setminus T \\ ext.active}} Y_{c(e)} \prod_{\substack{e \in T \\ int.inactive}} x_{c(e)} \prod_{\substack{e \in E \setminus T \\ ext.inactive}} y_{c(e)},$$

where the sum is over all spanning trees. To define the coloured Tutte polynomial for unconnected graphs, add indeterminates α_i, $i \in \mathbb{N}$, to the ring $\mathbb{Z}_\Lambda$ and call the resulting ring $\mathbb{Z}_{\Lambda,\alpha_i}$. Denote by $\mathcal{G}_c$ a class of edge coloured graphs with colours in Λ. The coloured Tutte polynomial is then a map

$$W : \mathcal{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i}$$

defined as

$$W(G, c, \phi) = \alpha_{k(G)} \prod_{i=1}^{k(G)} W(G_i, c, \phi),$$

where the G_i are the connected components of G. If there is no danger of confusion, write $W(G, c) = W(G, c, \phi)$, or $W_{G,c}$ if the emphasis is on the polynomial for a fixed graph rather than the map. Note that W is not invariant under graph isomorphism, since it depends on the colouring.

In contrast to the standard Tutte polynomial, this polynomial is not independent from the ordering of the edges, as the following example from [BR99] shows. Consider the graph I_2, consisting of two vertices and two parallel edges e_1 and e_2, with colours λ and μ respectively. Let ϕ be an ordering defined by $\phi(e_i) = i$ and ϕ' the other possible ordering of the edges. Then:

$$W_{I_2,c,\phi} = X_\lambda y_\mu + Y_\lambda x_\mu,$$
$$W_{I_2,c,\phi'} = x_\lambda Y_\mu + y_\lambda X_\mu.$$

So it is natural to ask for which *substitutions* of the indeterminates with values from some other ring R, the coloured Tutte polynomial gives an order independent map from the class $\mathcal{G}_c$ of coloured graphs (with colours in Λ) to the ring R. If $R = \mathbb{Z}[\overline{w}]$ is again a polynomial ring, then such a substitution can be described by a homomorphism. For example, by assigning $X_\lambda \mapsto x$, $Y_\lambda \mapsto y$, $x_\lambda, y_\lambda \mapsto 1$, $\lambda \in \Lambda$, $\alpha_i = 1$, an order (and colour) independent map from $\mathcal{G}_c$ to $\mathbb{Z}[x, y]$ is obtained, namely the classical Tutte polynomial.

Let us for now restrict attention to connected graphs, so that the α_i do not have to be considered. Let $\psi : \mathbb{Z}_\Lambda \to R$ be a substitution. If I is an ideal in $\mathbb{Z}_\Lambda$, such that $I \subseteq ker(\psi)$, then there is a unique homomorphism

$\overline{\psi} : \mathbb{Z}_\Lambda/I \to R$ so that ψ factors as $\psi = \overline{\psi} \circ p$, where p ist the projection $\mathbb{Z}_\Lambda \to \mathbb{Z}_\Lambda/I$.

Therefore, the related question can be asked for which ideals I the composition $\mathcal{G}_c \to \mathbb{Z}_\Lambda \to \mathbb{Z}_\Lambda/I$ is an order independent map. This is answered by the following proposition, due to Bollobás and Riordan [BR99].

Proposition 8. *(Bollobás, Riordan) Let $I \subseteq \mathbb{Z}_\Lambda$ be an ideal and denote by $\overline{W}$ the composition*

$$\overline{W} : \mathcal{G}_c \to \mathbb{Z}_\Lambda \to \mathbb{Z}_\Lambda/I.$$

Then, for any two orderings ϕ and ϕ', $\overline{W}(G, c, \phi) = \overline{W}(G, c, \phi')$ holds, if and only if $I_0 \subseteq I$, where I_0 is the minimal Ideal such that

$$\begin{aligned} X_\lambda y_\mu - y_\lambda X_\mu - x_\lambda Y_\mu + Y_\lambda x_\mu &\in I_0, \\ X_\nu(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) &\in I_0, \\ Y_\nu(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) &\in I_0 \end{aligned}$$

for any colours λ, μ and ν.

When considering disconnected graphs, Proposition 8 also holds for ideals $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$, with the condition that $I_0' \subseteq I$, where I_0' is the ideal in $\mathbb{Z}_{\Lambda,\alpha_i}$ generated by $\cup_i \alpha_i I_0$.

Under the same condition as in Proposition 8 of section 2.2, Bollobás and Riordan show that there is an interpretation of its values in terms of the cut and paste operations using $G - e$ and G/e. The colouring on the graphs $G - e$ and G/e is simply the induced colouring from c.

Proposition 9. *The coloured Tutte polynomial $W(E_n, c)$ defines a unique map $\mathcal{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ such that:*

$$W(E_n, c) = \alpha_n$$

$$W(G, c) = \begin{cases} X_{c(e)} W(G/e, c) & \text{if } e \text{ is a bridge} \\ Y_{c(e)} W(G - e, c) & \text{if } e \text{ is a loop} \\ x_{c(e)} W(G/e, c) + y_{c(e)} W(G - e, c) & \text{else.} \end{cases}$$

Here, E_n denotes the graph consisting of n isolated vertices. Lets now return to substitutions of the indeterminates with elements from a ring $R = \mathbb{Z}[w_1, w_2, \ldots]$. Consider again a homomorphism $\psi : \mathbb{Z}_{\Lambda,\alpha_i} \to R$. Write $W(G, c; \psi)$ for the composition

$$\overline{W} : \mathcal{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i} \to \mathbb{Z}[w_1, w_2, \ldots].$$

In order to be able to apply the recursion given by Proposition 9 to $W(G, c; \psi)$, we have to show that the result is independent of the ordering of the edges, i.e. it is sufficient to show that $I_0 \subseteq ker(\psi)$, which by the definition of I_0 is the case if and only if

$$\begin{aligned} \psi(X_\lambda y_\mu - X_\mu y_\lambda + Y_\lambda x_\mu - Y_\mu x_\lambda) &= 0, \\ \psi(X_\nu(x_\mu y_\lambda - x_\lambda y_\mu + Y_\mu x_\lambda - Y_\lambda x_\mu)) &= 0, \\ \psi(Y_\nu(x_\mu y_\lambda - x_\lambda y_\mu + Y_\mu x_\lambda - Y_\lambda x_\mu)) &= 0. \end{aligned} \tag{1}$$

This will prove to be a useful tool in showing that certain generating functions are indeed substitutions of the coloured Tutte polynomial. In some situations, the map ψ will be omitted and, for example, $X_\lambda = w_i$ will be written instead of $\psi(X_\lambda) = w_i$.

2.3. **How does the Tutte polynomial fit into VNP?** For what follows, the field underlying **VNP** will implicitly be assumed to be $\mathbb{R}$. In order to study the coloured Tutte polynomial in the framework of algebraic complexity theory, we have to choose a family of coloured and edge ordered graphs (G_n, c_n, ϕ_n). Not any family will do. The number of edges of the G_n should be polynomially bounded in n, and, in order to be able to construct a p-projection to a **VNP**−COMPLETE family, the number of edges as well as the cardinality of the range of the c_n should be functions polynomially bounded from below. For simplicity, consider maximal coloured, connected graphs, where maximal coloured means that each edge gets a different colour. A suitable family of graphs will be determined by the needs of the p-projection to be constructed. Also, the family (W_{G_n,c_n}) under consideration should be a member of **VNP**, and therefore p-definable. We use Valiant's criterion to show this. For similar arguments, we refer to the proof of Lemma 1 in [Ann95].

Theorem 10. *Let (G_n, c_n) be a family of edge-ordered, maximal coloured graphs, so that the corresponding family of coloured Tutte polynomials $W = (W_{G_n,c_n})$ is a p-family. Then, W is p-definable, and hence, $W \in$ **VNP**.*

Proof. For $n \in \mathbb{N}$, assume $E(G_n) = \{e_1, \ldots, e_m\}$, where $m = p(n)$ for a polynomial p. For a spanning tree T of G_n, define functions $\sigma^i : E(G_n) \to \{0,1\}^m$, $1 \le i \le 4$, given by:

$$\sigma^1(e) = \begin{cases} 1 & \text{e internally active} \\ 0 & \text{else} \end{cases}$$

$$\sigma^2(e) = \begin{cases} 1 & \text{e externally active} \\ 0 & \text{else} \end{cases}$$

$$\sigma^3(e) = 1 - \sigma^1(e)$$

$$\sigma^4(e) = 1 - \sigma^2(e)$$

Call the concatenation $f \in \{0,1\}^{4m}$ of the incidence vectors of the σ^i the activity vector of (G_n, T). Let ϕ_n be the function $\phi_n : \{0,1\}^{4m} \to \mathbb{N}$, such that $\phi_n(f)$ gives the number of spanning trees for which f is the activity vector. Note that ϕ can take values at most 1. Then, setting $(Z_1, \ldots, Z_{4m}) = (X_1, \ldots, X_m, Y_1, \ldots, Y_m, x_1, \ldots, x_m, y_1, \ldots, y_m)$, the polynomial W_{G_n,c_n} can be reformulated as follows:

$$W_{G_n,c_n} = \sum_{f \in \{0,1\}^{4m}} \phi_n(f) \prod_{i=1}^{4m} Z_i^{f_i}.$$

It remains to be seen, that ϕ_n is in **#P**.

This follows from the fact that the problem *is there a spanning tree T, such that f is the activity vector ?* is in **NP**. Valiant's criterion (Proposition 5) now shows that the p-family W is p-definable. □

2.4. **The Tutte polynomial in statistical mechanics.** To motivate the reduction we are going to construct for the coloured Tutte polynomial, we review the role of the Tutte polynomial in statistical mechanics. The Tutte polynomial appears in relation to the partition function defined on lattices.

Of interest is the role of the partition function in the Ising model. For a graph-theoretical overview of the Ising model, see [Kas67] or [Wel93b]. Consider a lattice interpreted as graph $G = (V, E)$, with an ordering of the edges. A map $\sigma : V(G) \mapsto \{+1, -1\}$, which assigns to each particle a spin, is called a *configuration* (or state) of G. Further, associate with each edge $e = \{i, j\}$ of G a weight, or interaction energy I_{ij}. In absence of an external field and boundary conditions, the Hamiltonian of a configuration is defined as $H(\sigma) = -\sum_{(ij)} I_{ij}\sigma_i\sigma_j$. The partition function is then defined as

$$Z_G^{Ising} = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where $\beta = 1/kT$, k is the Boltzmann constant and T the temperature. One physical interpretation is, that the probability of finding the system in a configuration σ is given by $e^{-\beta H(\sigma)}/Z_G^{Ising}$. Assume for now that $I_{ij} = I$ is constant on all edges. In this case, there is a known relationship between the partition function, the Tutte polynomial and the generating function for closed subgraphs. For a proof of the following assertions, see for example [Wel93b].

Proposition 11. *The partition function is related to the Tutte polynomial by the following equation:*

$$Z_G^{Ising}(\beta, I) = \left(2e^{-\beta I}\right)^{|E|-r(G)} (4sinh(\beta I))^{r(G)}\, T_G\left(coth(\beta I), e^{2\beta I}\right).$$

Further, if $C_G(x)$ denotes the generating function for closed subgraphs with equal weights, i.e.

$$C_G(x) = \sum_{\substack{E' \subseteq E \\ (V,E')closed}} x^{|E'|},$$

then the partition function can be written as

$$Z_G^{Ising}(\beta, I) = cosh(I\beta)^{|E|} 2^{|V|} C_G(-tanh(I\beta)).$$

The following expression for C_G in terms of the Tutte polynomial is now apparent:

Corollary 12. *The generating function C_G for constant weights can be written as*

$$C_G(x) = 2^{|E|-k(G)}(1+x)^{|E|}\left(-\frac{x}{1+x}\right)^{r(G)} T_G\left(\frac{-1}{x}, \frac{1-x}{1+x}\right)$$

3. Proof of Theorems 1 and 2

3.1. **The technical lemmas for Theorem 1.** The link between the maximal coloured Tutte polynomial and the generating function $C(G) = C_G$ will be a polynomial closely related to the partition function from statistical mechanics. This is the so-called dichromatic polynomial $Q(G)$, as introduced by Traldi in [Tra89]. Its relation to the coloured Tutte polynomial is also pointed out in [BR99].

We recall Definition 1 that for a graph $G = (V, E)$, the polynomial

$$Q(G; v_e, q, z) = \sum_{E' \subseteq E} \Big(\prod_{e \in E'} v_e\Big) q^{k\langle E'\rangle} z^{|E'|-r\langle E'\rangle}$$

is called the *dichromatic polynomial* of G.

There is no loss of generality in considering the restriction to $z = 1$, since

$$Q(G; v_e, q, z) = z^{-|V|} Q(G; zv_e, zq, 1).$$

From now on we write simply $Q(G; v_e, q) := Q(G; v_e, q, 1)$ and call this the dichromatic polynomial.

The main idea in the proof of Theorem 1 is to relate the dichromatic polynomial to both the Tutte polynomial and the generating function of closed subgraphs.

First, a characterization of $Q(G)$ in terms of the cut-and-paste operations using the graphs $G - e$ and G/e will prove to be very useful.

Lemma 13. *The polynomial $Q(G)$ is uniquely determined by the following relations:*

$$Q(E_n) = q^n,$$
$$Q(G) = Q(G - e) + v_e Q(G/e).$$

Proof. This proof is based on the proof of [Bol98, Theorem 4, chapter 10] for equal weights $v_e = v$. It is easy to see that $Q(E_n) = q^n$. Let $G = (V, E)$ be a graph and $f \in E$. Then:

$$\begin{aligned} Q(G) &= \sum_{E' \subseteq E \setminus \{f\}} \Big(\prod_{e \in E'} v_e \Big) q^{k\langle E' \rangle} + \sum_{\substack{E' \subseteq E \\ f \in E'}} \Big(\prod_{e \in E'} v_e \Big) q^{k\langle E' \rangle} \\ &= Q(G - f) + v_f \sum_{E' \subseteq E \setminus \{f\}} \Big(\prod_{e \in E'} v_e \Big) q^{k\langle E' \cup \{f\} \rangle} \\ &= Q(G - f) + v_f Q(G/f). \end{aligned}$$

It is clear, that the above relations uniquely determine the polynomial. □

Of importance is the case, where $q = 2$. Write $Z(G; v_e) := Q(G; v_e, 2)$ in this case. Using Lemma 13 the following characterisation of $Z(G; v_e)$, similar to one given in [Bol98, Chapter 10], can be given.

Lemma 14. *Let $G = (V, E)$ be an edge-weighted graph with weights $v_e, e \in E$. Then:*

$$Z(G; v_e) = \sum_{\sigma \in \{-1,+1\}^{|V|}} \prod_{e=(i,j) \in E} (1 + v_e)^{\frac{\sigma_i \sigma_j + 1}{2}},$$

Proof. Set

$$\tilde{Z}(G) = \sum_{\sigma} \prod_{e=\{i,j\} \in E} (1 + \delta(\sigma_i, \sigma_j) v_e).$$

Since the σ_i only take values in $\{-1, +1\}$, the identity $(1 + v_e)^{\frac{\sigma_i \sigma_j + 1}{2}} = 1 + \delta(\sigma_i, \sigma_j) v_e$ holds, where δ is the Kronecker function, i.e. $\delta(a, b) = 1$ if $a = b$ and 0 otherwise. Thus, $\tilde{Z}_G$ is just the right hand side of equation of Lemma 14.

Checking that $\tilde{Z}(E_n) = 2^n$ is straightforward. Let $f = (l, k)$ be an edge in E. Then:

$$\begin{aligned}
\tilde{Z}(G) &= \sum_{\substack{\sigma \\ \sigma_l \neq \sigma_k}} \prod_e (1 + \delta(\sigma_i, \sigma_j) v_e) + \sum_{\substack{\sigma \\ \sigma_l = \sigma_k}} \prod_e (1 + \delta(\sigma_i, \sigma_j) v_e) \\
&= \sum_{\sigma} \prod_{e \in E \setminus \{f\}} (1 + \delta(\sigma_i, \sigma_j) v_e) + v_f \sum_{\substack{\sigma \\ \sigma_l = \sigma_k}} \prod_{e \in E \setminus \{f\}} (1 + \delta(\sigma_i, \sigma_j) v_e) \\
&= \tilde{Z}(G - e) + v_f \tilde{Z}(G/e).
\end{aligned}$$

This shows that $\tilde{Z}(G)$ satisfies the equation of Lemma 13 with $q = 2$ and is therefore equal to $Z(G)$. □

By setting $v_e = e^{2\beta I_{ij}} - 1$, the partition function from the Ising model is obtained:

$$Z(G) = \Big(\prod_{e \in E} e^{\beta I_e}\Big) Z_G^{Ising}.$$

Next we establish, in analogy to the partition function for constant weights $I_{ij} = I$, as given in Corollary 12 of section 2.4, the following relationship between the dichromatic polynomial and the generating function for closed subgraphs.

Lemma 15. *For edge-weighted graphs $G = (V, E)$ and $C(G)$ as above, the following holds:*

$$C(G; w_e) = \Big(\prod_{e \in E} (1 - w_e)\Big) Z\Big(G; \frac{2w_e}{1 - w_e}\Big).$$

Proof. This proof follows closely the arguments given in [Wel93b, Page 61]. Before going into the calculation, consider the function

$$\sum_{E' \subseteq E} \sum_{\sigma \in \{-1,+1\}^{|V|}} \prod_{e \in E'} \sigma_i \sigma_j x_e,$$

where i, j are always assumed to be the end-vertices of e in the product. If (V, E') is closed, then every σ_i occurs an even number of times in the product. In this case, $\prod_{e \in E'} \sigma_i \sigma_j x_e = \prod_{e \in E'} x_e$. Suppose (V, E') is not closed. Then, there is a k, such that σ_k occurs an odd number of times in the product. Therefore:

$$\sum_{\sigma} \prod_{e \in E'} \sigma_i \sigma_j x_e = \sum_{\substack{\sigma \\ \sigma_k = +1}} \prod_{e \in E'} \sigma_i \sigma_j x_e + \sum_{\substack{\sigma \\ \sigma_k = -1}} \prod_{e \in E'} \sigma_i \sigma_j x_e = 0.$$

Now, the claim follows from the following observations, where the first step uses Lemma 14. Set $v_e = \frac{2w_e}{1-w_e}$.

$$\begin{aligned}
Z(G; v_e) &= \sum_{\sigma} \prod_{e\in E} (1+v_e)^{\frac{\sigma_i\sigma_j+1}{2}} \\
&= \prod_{e\in E}\Big(\frac{v_e+2}{2}\Big) \sum_{\sigma}\prod_{e\in E}\Big(\sigma_i\sigma_j\frac{v_e}{v_e+2}+1\Big) \\
&= \prod_{e\in E}\Big(\frac{v_e+2}{2}\Big) \sum_{E'\subseteq E}\sum_{\sigma}\prod_{e\in E'}\Big(\sigma_i\sigma_j\frac{v_e}{v_e+2}\Big) \\
&= \prod_{e\in E}\Big(\frac{v_e+2}{2}\Big) \sum_{\substack{E'\subseteq E \\ (V,E')closed}}\prod_{e\in E'}\Big(\frac{v_e}{v_e+2}\Big) \\
&= \Big(\prod_{e\in E}\frac{1}{(1-w_e)}\Big) C(G; w_e)
\end{aligned}$$

□

This gives the relation between $Z(G)$ and $C(G)$. Next, what is needed is a relation between the coloured Tutte polynomial and $Z(G)$. The following lemma accomplishes this:

Lemma 16. *Let $G = (V, E)$ be an edge-ordered, maximal coloured graph. Then, under the substitution*

$$X_{c(e)} = 2 + v_e, \quad Y_{c(e)} = 1 + v_e, \quad x_{c(e)} = v_e, \quad y_{c(e)} = 1.$$

for each $e \in E$, the following identity holds:

$$Z(G; v_e) = W(G; X_{c(e)}, Y_{c(e)}, x_{c(e)}, y_{c(e)}).$$

In other words, for any maximal coloured family of graphs G_n, so that W_{G_n} is a p-family, the family Z_{G_n} is a substitution of the family W_{G_n}.

Proof. First check that the maximal coloured Tutte polynomial is order-independent under the above substitutions. This follows by using the identities 1 after Proposition 9 of section 2.2:

$$\begin{aligned}
(2+v_e) - (2+v_f) - v_e(1+v_f) + (1+v_e)v_f &= 0 \\
v_e(1+v_f) - (1+v_e)v_f - v_e + v_f &= 0.
\end{aligned}$$

Therefore, $W(G, c)$ is order independent. The recursive definition 9 can now be applied. Set $\alpha_n = 2^n$. Then:

$$W(E_n) = 2^n$$

$$W(G) = \begin{cases} (2+v_e)W(G/e) & \text{if e is a bridge} \\ (1+v_e)W(G-e) & \text{if e is a loop} \\ W(G-e) + v_e W(G/e) & \text{else} \end{cases}$$

Clearly, if e is a loop, then $W(G-e) = W(G/e)$ and $W(G) = W(G-e) + v_e W(G/e)$. Suppose e is a bridge. Then $G-e$ has the same number of spanning forests as G/e, and an edge is internally (externally) active in $G-e$ if an only if it is internally (externally) active in G/e. But, since $G-e$ has one more component than G/e, with the choice of the α_n it follows that

$2W(G/e) = W(G-e)$. Therefore, again $W(G-e) + v_e W(G/e) = W_G$ is obtained. This coincides exactly with the recursion for $Z(G)$. □

Theorem 1 now follows by an easy computation combining lemma 15 with lemma 16:

Proof of Theorem 1. From the above considerations it follows that

$$\begin{aligned} C_G(w_e) &= \Big(\prod_{e\in E}(1-w_e)\Big) Z_G\Big(\frac{2w_e}{1-w_e}\Big) \\ &= \Big(\prod_{e\in E}(1-w_e)\Big) W_{G,c}\Big(\frac{2}{1-w_e}, \frac{1+w_e}{1-w_e}, \frac{2w_e}{1-w_e}, \frac{1-w_e}{1-w_e}\Big) \\ &= W_{G,c}\Big(2, 1+w_e, 2w_e, 1-w_e\Big). \end{aligned}$$

Here the first step is Lemma 14, and the second step uses Lemma 16 for the second term in the product. Again, order independence can be checked. □

Note that C_G can be written as

$$C_G(w_e) = \sum_{E'\subset E}\left(\prod_{e\in E'} w_e\right)\left(\prod_{e\in E-E'}(1-w_e)\right) 2^{k\langle E'\rangle + |E'|},$$

a form that closely resembles the so called probability generating functions as considered in [Bür00a].

3.2. **Invariance under edge replacement.** Before we are able to prove Theorem 2, we need some further results. Our approach to obtain a p-projection is as follows:

(1) Replace each edge e in C_n by edges in series with new colours.
(2) Assign values to the variables of the new colours that are constants or w_e.
(3) Make sure that under this substitution, $W_{C'_n}$ is the same as W_{C_n} with the substitutions from Theorem 1.

The following lemma gives conditions under which the Tutte polynomial is invariant under replacing two edges in series with a single one.

Lemma 17. *Let $G = (V, E)$ be a coloured graph with ordering ϕ and $e \in E$ an edge with colour $c(e) = \lambda$. Call G' the graph after replacing e with two edges e_1 and e_2 in series, with colouring $c(e_1) = \nu$, $c(e_2) = \mu$, and ordering ϕ', so that $\phi'(e_1) = \phi(e)$, $\phi'(e_2) = \phi(e)+1$, $\phi'(f) = \phi(f)+1$ if f is an edge with $\phi(f) > \phi(e)$ and $\phi = \phi'$ else. If $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ is an ideal such that*

$$\begin{aligned} X_\lambda - X_\nu X_\mu &\in I, \\ Y_\lambda - Y_\nu x_\mu - X_\nu y_\mu &\in I, \\ x_\lambda - x_\nu x_\mu &\in I, \\ y_\lambda - y_\nu x_\mu - X_\nu y_\mu &\in I, \end{aligned}$$

then $W(G', c', \phi') = W(G, c, \phi)$.

The statement of [BR99, Theorem 9] is the same as above but under the additional assumption that $I_0 \subseteq I$, i.e., preserving order independence. The problem is that there seems to be no way of making such substitutions in our case and at the same time preserving order independence. We now show how to proceed without relying on this assumption.

Recall that it was crucial that the replacement of $(X_{c(e)}, Y_{c(e)}, x_{c(e)}, y_{c(e)})$ by $(2, 1+w_e, 2w_e, 1-w_e)$ in $W_{C_n,c}$ gives a polynomial in the w_e independent of the order of the edges, since under this replacement, $W_{C_n,c}$ equals C_{C_n} and C_G is always order independent. But when changing to a different graph C'_n, such that $C_{C_n}(w_e) = W_{C'_n,c';\phi}$ under some replacement ϕ, then $W_{C'_n,c';\phi}$ does not have to be independent of the order in C'_n anymore. This is simply because $W_{C'_n,c'}$ does not depend on the same graph as C_{C_n}.
The drawback of Lemma 17 as stated here is that the proof is more involved, since the relations of Proposition 8 cannot be used.

Proof of Lemma 17. Replace the edge e with colour λ by two edges e_1, e_2 in series, with colours ν and μ respectively. Any spanning tree T going through e in G corresponds to a spanning tree T' going through both e_1 and e_2 in G', and which coincides with T on $E\backslash\{e\}$. Also, any spanning tree that does not traverse e in G corresponds to two spanning trees T'_1 and T'_2 in G', with $e_1 \notin T'_1$ and $e_2 \notin T'_2$. The idea of the proof is to show that the monomial in $W_{G,c,\phi}$ corresponding to a tree T in G is equal to the monomial in $W_{G',c',\phi'}$ corresponding either to the tree T' or to the sum of the monomials corresponding to the trees T'_1 and T'_2, depending on whether T goes through e or not. Notice that in any case, the contribution of the edges in $E\backslash\{e\}$ to a monomial will be the same as the contribution of the edges in $E'\backslash\{e_1, e_2\}$. Fix a spanning tree T in G.

(1) Assume the spanning tree T goes through e in G. Then e is either internally active or internally inactive.

Let e be internally active. Then, both e_1 and e_2 are internally active with respect to T' in G'. The contribution of e_1 and e_2 to the monomial of T' in $W_{G',c',\phi'}$ is then $X_\nu X_\mu$, which in $\mathbb{Z}_{\Lambda,\alpha_i}/I$ equals X_λ, the contribution of e to the monomial of T in $W_{G,c,\phi}$.

Let e be internally inactive. Then, both e_1 and e_2 are internally inactive with respect to T' in G' and their contribution to the corresponding monomial is $x_\nu x_\mu$. The monomials for T' and T are again equal in $\mathbb{Z}_{\Lambda,\alpha_i}/I$.

(2) Assume the spanning tree T does not go through e in G. Then e is either externally active or externally inactive.

Let e be externally active. Then, e_1 will be externally active with respect to T'_1 in G' and e_2 internally active. This is because the only other edge in the cut defined by e_2 is e_1, which beats e_2 because of its priority in the ordering. The contribution to the corresponding monomial is thus $Y_\nu x_\mu$. With respect to T'_2, similar arguments show that e_1 is internally active and e_2 externally inactive. This gives a contribution of $X_\nu y_\mu$ to the monomial of T'_2. Recall that the monomials for T'_1 and T'_2 coincide for edges other than e_1 and e_2. Therefore, the sum of those monomials will be same as the monomial for T in G, with Y_λ replaced by $Y_\nu x_\mu + X_\nu y_\mu$. But those two terms are equal in $\mathbb{Z}_{\Lambda,\alpha_i}/I$.

Let e be externally inactive. Basically the same arguments as above show, that with respect to T'_1, e_1 is externally inactive and

e_2 internally inactive, giving a contribution of $y_\nu x_\mu$. As of T_2', e_1 is again internally active and e_2 externally inactive, contributing $X_\nu y_\mu$. So the sum of the monomials for T_1' and T_2' equals the monomial for T in the quotient ring.

This completes the proof. □

3.3. **Proof of Theorem 2.** Take a maximally coloured graph C_n, enumerate the edges $e_1, \ldots, e_m$ and set $(X_{c(e)}, Y_{c(e)}, x_{c(e)}, y_{c(e)}) = (2, 1+w_e, 2w_e, 1-w_e)$ for each edge e. Now replace each edge e_i by two edges e_i^1 and e_i^2 in series and apply a new ordering $\phi'(e_i^1) = 2i-1$, $\phi'(e_i^2) = 2i$. Also assign to the edges new colourings $c'(e_i^1) = \nu_i$ and $c'(e_i^2) = \mu_i$. Then, under the assignment

$$\begin{array}{llll} X_{\nu_i} = 2 & X_{\mu_i} = 1 & Y_{\nu_i} = 1 & Y_{\mu_i} = 1 \\ x_{\nu_i} = 2 & x_{\mu_i} = w_{e_i} & y_{\nu_i} = -1 & y_{\mu_i} = 1/2 \end{array}$$

the relations of Lemma 17 are satisfied and $C_{C_n,c}$ is a projection of $W_{C_n',c',\phi'}$. So we apply Proposition 7, which states that $C_{C_n,c}$ is **VNP**−COMPLETE, hence $W_{C_n',c',\phi'}$ is **VNP**−COMPLETE. □

Note that even though $W_{C_n',c'}$, with the above substitution, depends on the order chosen for C_n', as long as the ordering is constructed as in the proof of Theorem 2 from *any* ordering of C_n, this substitution will give the desired result.

4. Conclusions and further research

We have discussed the worst case complexity of the weighted and colored Tutte polynomials in the framework of Valiant's model of non-uniform algebraic computation. We have shown that for suitable chosen families of graphs G_n the corresponding families of coloured Tutte polynomials are **VNP**-complete with respect to p-projections. The same is true for the weighted Tutte polynomial, if we allow oracle reductions. We have already asked, Problem 1 in the introduction, whether p-projections suffice.

In both cases, coloured or weighted, it is essential that the number of variables of the Tutte polynomial grows linearly in the number of edges of the graph. The following is now a natural question.

Problem 2. Are there sparse graphs G_n such that the corresponding family of coloured Tutte polynomials are still **VNP**−COMPLETE?

In Corollary 3 we have seen that the permanent PER_n and the various variations of the Tutte polynomial can be obtained from each other by substitution instances.

Problem 3. Find a natural explicit description and graph theoretic interpretation of the substitutions which produces from the, say, coloured Tutte polynomial the permanent (and vice versa).

So far we have discussed the complexity of the Tutte polynomial in the *non-uniform* setting of algebraic circuits. In practice the examples studied are, however, uniformly computable. Another approach to the complexity of the Tutte polynomial and its relatives would be to use the uniform algebraic model popularized by L. Blum, M. Shub and S. Smale [BCSS98] and now commonly called the BSS-model of computation. For the long history of this computational model the reader should consult the encyclopedic article by J. Tucker and J. Zucker [TZ00].

There has been a generalization of **#P** in the BSS-model, due to K. Meer [Mee00]. The problem with this generalization is that it is based on counting functions and does not capture the combinatorial aspects of generating functions of graph properties. An attempt at introducing suitable complexity classes for such functions has been proposed by J. Makowsky and K. Meer [MM00].

Problem 4. What is the right uniform algebraic context in which to study the Tutte polynomial? Is it, or its coloured variant, complete in a class related to the BSS-model of computation?

The problem is, that there seems to be no naturally maximal class of functions, for which one can try to prove completeness.

Finally, we have mentioned in the introduction briefly the issue of approximability of the Tutte polynomial in the Turing model of computation. For a recent survey of approximability in general, cf. the book of Hochbaum [Hoc97, Chapters 9-12]. In particular, $FPRAS$ approximability was established for Tutte polynomials on dense graphs by Alon, Frieze and Welsh [AFW94, AFW95] and improved by Karger [Kar99] and for permanents of matrices with non-negative entries by Jerrum, Sinclair and Vigoda, [JSV00]. These results are definitely interesting and meaningful, especially when the approximation of the value of the polynomial to be computed has a natural interpretation, such as probabilities of network reliability in [Kar99], or when the approximability has been an outstanding question which resisted many attempts, such as the approximability of the permanent, [JSV00].

Clearly, approximability is based on notions of *nearness* such as provided in an ordered ring, and *randomness* such as provided by some coin-tossing mechanism. However, this approach has, to the best of our knowledge, not yet been thoroughly studied neither in Valiant's model nor in the BSS model. So the following remains a challenge:

Problem 5. Create a meaningful framework for *Fully Polynomial Randomized Approximations Schemes* $FPRAS$ in Valiant's model over ordered rings and to study the approximability of the various Tutte polynomials in this framework.

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